Assignment 7

Ross Tate

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Exercise 1. Given a diagram $D: \mathbf{I} \to \mathbf{C}$ where \mathbf{I} has a terminal object 1 and \mathbf{C} has pullbacks, show that for every morphism $f: A \to D(1)$ there exists a diagram $D_f: \mathbf{I} \to \mathbf{C}$ and a collection of morphisms $\{f_i: D_f(i) \to D(i)\}_{i \in \mathbf{I}}$ such that f_1 equals f and for every morphism $m: i \to j$ in \mathbf{I} the following is a pullback:

$$D_f(i) \xrightarrow{f_i} D(i)$$

$$D_f(m) \downarrow \qquad \qquad \downarrow D(m)$$

$$D_f(j) \xrightarrow{f_i} D(j)$$

To save a step in the proof, utilize Proposition 11.10(2) of Abstract and Concrete Categories.

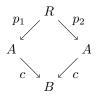
Remark. The above process can in particular be used to construct the pullback of any finite cocone. Thus, just like there is a notion of pullback stability for classes of morphisms, there is also a notion of pullback stability for classes of cocones.

Definition. A pair of morphisms $p_1, p_2 : R \to A$ are a kernel pair if there exists a morphism $f : A \to B$ such that the following is a pullback:



Definition. A triple $\langle p_1 : R \to A, p_2 : R \to A, c : A \to B \rangle$ is an exact sequence when $\langle p_1, p_2 \rangle$ is a kernal pair and c is a coequalizer of $\langle p_1, p_2 \rangle$.

Exercise 2. Show that for any coequalizer c of any kernel pair $\langle p_1, p_2 \rangle$, the following is a pullback



In other words, $\langle p_1, p_2 \rangle$ is specifically a kernel pair of c.

Exercise 3. Show that the underlying functor of Alg(2) reflects coequalizers, meaning that if |c| is a coequalizer of $|f_1|$ and $|f_2|$ in Set then c is a coequalizer of f_1 and f_2 in Alg(2).

Exercise 4. Without using quotient types or sets of equivalence classes, show that every kernel pair in Alg(2) has a coequalizer and, using this construction, that the underlying functor of Alg(2) preserves exact sequences.

Definition. A category is regular if and only if it is a finitely complete category such that exact sequences are pullback stable and every kernel pair has a coequalizer (i.e. is part of some exact sequence).

Exercise 5. Show that Alg(2) is a regular category (understanding that we have already proven in class that Alg(2) is finitely complete). Hint: take advantage of the fact that **Set** is regular.