Weighted Limits and Colimits

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**Definition** (Comma Object). Given 1-cells $A_1 \xrightarrow{f_1} B \xleftarrow{f_2} A_2$ of a 2-category, a comma object from $f_1$ to $f_2$ is a 0-cell, typically denoted $f_1 \downarrow f_2$, along with 1- and 2-cells as in the following diagram

\[
\begin{array}{c}
\pi_2 & \xrightarrow{f_2} & A_2 \\
\downarrow & & \downarrow \\
\pi_1 & \xrightarrow{f_1} & A_1 \\
\end{array}
\]

\[
\begin{array}{c}
p_2 & \xrightarrow{A_2} & \xrightarrow{f_2} B \\
\downarrow & & \downarrow \\
p_1 & \xrightarrow{A_1} & \xrightarrow{f_1} B \\
\end{array}
\]

that is *universal* in the sense that given any other diagram $C \xleftarrow{\alpha} B$ there exists a unique 1-cell $\langle \alpha \rangle : C \rightarrow f_1 \downarrow f_2$ such that $\langle \alpha \rangle \ast \pi_1$ equals $\alpha$ (and $\langle \alpha \rangle ; \pi_1$ equals $p_1$ and $\langle \alpha \rangle ; \pi_2$ equals $p_2$).

**Example.** Comma categories are the comma objects of $\text{Cat}$.

**Definition** (Ccomma Object). Given 1-cells $B_1 \xleftarrow{f_1} A \xrightarrow{f_2} B_2$ of a 2-category, a cocomma object from $f_1$ to $f_2$ is a 0-cell, typically denoted $f_1 \uparrow f_2$, along with 1- and 2-cells as in the following diagram

\[
\begin{array}{c}
f_2 & \xrightarrow{B_2} & \xleftarrow{\kappa_2} B_2 \\
\downarrow & & \downarrow \\
f_1 & \xrightarrow{B_1} & \xleftarrow{\kappa_1} B_1 \\
\end{array}
\]

\[
\begin{array}{c}
f_2 \uparrow B_2 & \xrightarrow{c_2} C \xleftarrow{f_1 \uparrow f_2} B_1 \\
\downarrow & & \downarrow \\
f_1 & \xleftarrow{f_1} & \xleftarrow{\kappa_1} B_1 \\
\end{array}
\]

that is *universal* in the sense that given any other diagram $A \xrightarrow{\alpha} C$ there exists a unique 1-cell $\langle \alpha \rangle : f_1 \uparrow f_2 \rightarrow C$ such that $\kappa_1 \ast [\alpha]$ equals $\alpha$ (and $\kappa_1 ; [\alpha]$ equals $c_1$ and $\kappa_2 ; [\alpha]$ equals $c_2$).

**Example.** For the 1-source $1 \xleftarrow{a} A \xrightarrow{id} A$ in $\text{Prost}$, the corresponding cocomma object $! \uparrow A$ is the set $\text{Option}(A)$ with none being smaller than some($a$) for all $a \in A$. On the flipside, the cocomma object $A \uparrow !$ is the set $\text{Option}(A)$ with none being larger than some($a$) for all $a \in A$. In both cases, some($a$) is less than some($a'$) iff $a$ is less than $a'$.

Note that $L(A)$ in $\text{Set}$ can be defined as the fixpoint $\mu X.1 + (A \times X)$. In $\text{Prost}$, the fixpoints $\mu X.1 + (A \times X)$, $\mu X.!(A \times X)$, and $\mu X.(A \times X) \uparrow !$ all correspond to lists but with different orderings. In the first, $\ell \leq \ell'$ can only hold if $\ell$ and $\ell'$ have the same length, whereas in the second $\ell$ can be a prefix of $\ell'$, and in the third $\ell'$ can be a prefix of $\ell$. In particular, they all agree on lists with the same length, in which case they use componentwise comparison; where they differ is how they handle lists of differing length.

**Definition** (Inserter Object). Given two 1-cells $f_1, f_2 : A \rightarrow B$ of a 2-category, an inserter from $f_1$ to $f_2$ is a 0-cell, typically denoted $\text{Ins}(f_1, f_2)$, along with 1-cell $\pi : \text{Ins}(f_1, f_2) \rightarrow A$ and 2-cell $\pi_\text{Ins} : \pi ; f_1 \Rightarrow \pi ; f_2 : \text{Ins}(f_1, f_2) \rightarrow B$ that is *universal*, meaning given any other 0-cell $C$ with 1-cell $f : C \rightarrow A$ and 2-cell $\alpha : f ; f_1 \Rightarrow f ; f_2 : C \rightarrow B$ there exists a unique 1-cell $\langle \alpha \rangle : C \rightarrow \text{Ins}(f_1, f_2)$ such that $\langle \alpha \rangle ; \pi$ equals $f$ and $\langle \alpha \rangle \ast \pi_\text{Ins}$ equals $\alpha$.

**Example.** Given an endofunctor $T : C \rightarrow C$, the category $\text{Alg}(T)$ is the inserter from $T$ to $\text{Id}_C$, and the category $\text{Coalg}(T)$ is the inserter from $\text{Id}_C$ to $T$. 


Definition (Coinserter Object). Given two 1-cells \( f_1, f_2 : A \to B \) of a 2-category, a coinserter from \( f_1 \) to \( f_2 \) is a 0-cell, \( \text{Coins}(f_1, f_2) \), along with 1-cell \( \kappa : B \to \text{Coins}(f_1, f_2) \) and 2-cell \( \kappa_{\text{Coins}} : f_1 \Rightarrow f_2 ; \kappa : A \to \text{Coins}(f_1, f_2) \) that is (co)universal, meaning given any other 0-cell \( C \) with 1-cell \( f : B \to C \) and 2-cell \( \alpha : f_1 ; f \Rightarrow f_2 ; f : A \to C \) there exists a unique 1-cell \( [\alpha] : \text{Coins}(f_1, f_2) \to C \) such that \( \kappa ; [\alpha] \) equals \( f \) and \( \kappa_{\text{Coins}} \ast [\alpha] \) equals \( \alpha \).

Definition (Weighted Limit). Let \( I \) be a 2-category conceptually describing a scheme, and let \( D : I \to C \) be a 2-functor conceptually describing a diagram of scheme \( I \) in the 2-category \( C \). Furthermore, let \( W : I \to \text{Cat} \) be a 2-functor conceptually describing a weighting of the diagram. A \( W \)-weighted cone of the diagram \( D \), denoted \( \text{lim} W D \), is a 0-cell \( L \) of \( C \) and a collection of 1-cells \( \{ \pi_w : L \to D I \}_{w \in W I} \) and 2-cells \( \{ \pi_w \ast \omega : \pi_w ; \omega \Rightarrow \pi_w ; \omega' \}_{w \in W I} \) that preserves identities and compositions, meaning \( \pi_{id_w} = id_{\pi_w} \) and \( \pi_{w ; \omega} = \pi_w ; \pi_{\omega'} \), and is natural, meaning for all 1-cells \( i : I \to I' \in I \) each appropriate 1-cell \( \pi_{W(i)} \) equals \( \pi_w \ast Di \) and for all 2-cells \( \iota : i \Rightarrow i' \in I \) each appropriate 2-cell \( \pi_{W(i)} \) equals \( \pi_w \ast Di \). A \( W \)-weighted limit of a diagram \( D \) is a universal \( W \)-weighted cone of \( D \).

Example. An inserter is a weighted limit as illustrated below:

\[
\begin{array}{c}
\begin{array}{c}
\text{I} \\
\bullet \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{C} \\
A \\
\begin{array}{c}
f_1 \\
f_2 \\
B
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Cat} \\
* \\
\begin{array}{c}
\cdots \to 1 \\
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Ins}(f_1, f_2) \\
\pi_1 = \pi_* : f_1 \\
\pi_2 = \pi_* : f_2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\pi_{\text{Ins}} \\
\end{array}
\end{array}
\end{array}
\end{array}
\]

Example. A comma object is a weighted limit. The scheme is \( \bullet \to \bullet \leftarrow \bullet \) and the weighting is \( 1 \to \left( 1 \to 2 \right) \leftarrow 2 \).

Definition. A weighted colimit is dual to a weighted limit: given a diagram \( D : I \to C \) and weighting \( W : I^{\text{op}} \to \text{Cat} \) (the reason that \( W \) is contravariant here is complicated and very meta), a weighted colimit \( \text{colim}_W D \) in \( C \) is a weighted limit \( \text{lim}_{W^{\text{op}}} D^{\text{op}} \) in \( C^{\text{op}} \).

Example. Coinserter and cocomma objects are weighted colimits of the same weighting but on the opposite scheme as for inserter and comma objects.