Reflective and Coreflective Subcategories

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1 Reflective Subcategories

Definition. **SymLMet** is the full category of **LMet** comprised of *symmetric* Lawvere metric spaces $\langle X, d\rangle$, meaning the following holds:

$$\forall x, x' \in X.\ d(x, x') = d(x', x)$$

Similarly, $\mathbf{SymLMet}_u$ and $\mathbf{SymLMet}_c$ are the corresponding full subcategories of $\mathbf{LMet}_u$ and $\mathbf{LMet}_c$.

Example. Given a Lawvere metric space $\langle X, d\rangle$, define the distance function $d_{\min}$ on X as follows:

$$d_{\min}(x, x') = \inf_{n\in\mathbb{N}, x_0,\ldots,x_n\in X} \min\{d(x, x_0), d(x_0, x)\} + \cdots + \min\{d(x_n, x'), d(x', x_n)\}$$

The Lawvere metric space $\langle X, d_{\min}\rangle$ is symmetric, and the identity function is a **SymLMet**-reflection arrow from $\langle X, d\rangle$ to $\langle X, d_{\min}\rangle$.

Definition. **SSLMet** is the full category of **SymLMet** comprised of symmetric, *separated* Lawvere metric spaces $\langle X, d\rangle$, meaning the following holds:

$$\forall x, x' \in X.\ d(x, x') = 0 \implies x = x'$$

Similarly, $\mathbf{SSLMet}_u$ and $\mathbf{SSLMet}_c$ are the corresponding full subcategories of $\mathbf{SymLMet}_u$ and $\mathbf{SymLMet}_c$.

Example. Given a symmetric Lawvere metric space $\langle X, d\rangle$, define the equivalence relation $\approx_0$ on X as

$$x \approx_0 x' \iff d(x, x') = 0$$

Then define the (provably well-defined due to symmetry and triangle inequality) distance function d_0 on ${}^{X}/_{\approx_0}$ as

$$d_0([x]_{\approx_0}, [x']_{\approx_0}) = d(x, x')$$

The Lawvere metric space $\langle X, d_0\rangle$ is symmetric and separated, and the function $\lambda x.\,[x]_{\approx_0}$ is a **SSLMet**-reflection arrow from $\langle X, d\rangle$ to $\langle {}^{X}/_{\approx_0}, d_0\rangle$.

2 Weakly Reflective Subcategory

Definition. Given categories $\mathbf{A} \subseteq \mathbf{B}$, an object $A \in \mathbf{A}$, and an object $B \in \mathbf{B}$, a $\mathbf{B}$-morphism $r : B \to A$ is a *weak* $\mathbf{A}$-reflection arrow for B if for every object $A' \in \mathbf{A}$ and $\mathbf{B}$-morphism $f : B \to A'$ there exists a (not necessarily unique) $\mathbf{A}$-morphism $f' : A \to A'$ such that $r\,;f'$ equals f.

Definition. A subcategory $\mathbf{A} \subseteq \mathbf{B}$ is *weakly* reflective if for every object $B \in \mathbf{B}$ there exists a *weak* $\mathbf{A}$-reflection arrow for B.

Definition. Given a symmetric Lawvere metric space $\langle X, d\rangle$, a point $x \in X$ is said to be a limit of a sequence $(x_i)_{i\in\mathbb{N}}$ of points in X if it satisfies the following property:

$$\forall \varepsilon \in \mathbb{R}^{>}.\ \exists n \in \mathbb{N}.\ \forall i \in \mathbb{N}.\ i > n \implies d(x_i, x) < \varepsilon$$

Any two limits of a sequence are guaranteed to be distance 0 apart from each other and so essentially equivalent. Thus one often speaks of "the" limit of a sequence and uses the notation $\lim_{i\to\infty} x_i$. Note that every sequence that has a limit is necessarily Cauchy.

Definition. A symmetric Lawvere metric space $\langle X, d\rangle$ is said to be (Cauchy) complete if every Cauchy sequence of $\langle X, d\rangle$ has a limit in $\langle X, d\rangle$.

Definition. **CSymLMet** is the (provably full) subcategory of **SymLMet** comprised of (Cauchy-)complete symmetric separated Lawvere metric spaces and limit-preserving morphisms. Similarly, $\mathbf{CSymLMet}_u$ and $\mathbf{CSymLMet}_c$ are the corresponding subcategories of $\mathbf{SymLMet}_u$ and $\mathbf{SymLMet}_c$, though only the former is provably full.

Definition. **CSSLMet** is the subcategory of **SSLMet** comprised of (Cauchy-)complete symmetric separated Lawvere metric spaces and limit-preserving morphisms. Similarly, $\mathbf{CSSLMet}_u$ and $\mathbf{CSSLMet}_c$ are the corresponding full subcategories of $\mathbf{SSLMet}_u$ and $\mathbf{SSLMet}_c$. These subcategories can all be proven to be full.

Definition. Given a symmetric Lawvere metric space $\langle X, d\rangle$, one can define a distance function d_ω betweens sequences $(x_i)_{i\in\mathbb{N}}$ and $(x'_i)_{i\in\mathbb{N}}$ of points in X (where inf and sup denote infimum and supremum, i.e. meet and join):

$$d_\omega((x_i)_{i\in\mathbb{N}}, (x'_i)_{i\in\mathbb{N}}) = \inf_{n\in\mathbb{N}} \sup_{i,i'\in\mathbb{N}>n} d(x_i, x'_i)$$

Definition. Given a symmetric Lawvere metric space $\langle X, d\rangle$, a sequence $(x_i)_{i\in\mathbb{N}}$ of points in X is said to be *Cauchy* if it satisfies the following property:

$$\forall \varepsilon \in \mathbb{R}^{>}.\ \exists n \in \mathbb{N}.\ \forall i, i' \in \mathbb{N}.\ i > n \wedge i' > n \implies d(x_i, x'_i) < \varepsilon$$

This is equivalent to the property that $d_\omega((x_i)_{i\in\mathbb{N}}, (x'_i)_{i\in\mathbb{N}}) = 0$.

Definition. Given a symmetric Lawvere metric space $\langle X, d\rangle$, its (Cauchy) completion $\overline{\langle X, d\rangle}$ is the symmetric Lawvere metric space comprised of the set of Cauchy sequences of $\langle X, d\rangle$ and the distance function d_ω. Assuming countable choice or excluded middle, $\overline{\langle X, d\rangle}$ is (Cauchy) complete, and (assuming arbitrary choice) the function $\lambda x.\,(x)_{i\in\mathbb{N}}$ is a *weak* reflection arrow in **SymLMet** into **CSymLMet** (and likewise for $\mathbf{SymLMet}_u$ into $\mathbf{CSymLMet}_u$ but *not* for $\mathbf{SymLMet}_c$ into $\mathbf{CSymLMet}_c$). Composing this weak reflection arrow with the reflection arrow from **SymLMet** into **SSLMet** results in a (strong) reflection arrow from **SymLMet** into **CSSLMet** (and likewise for $\mathbf{SymLMet}_u$ into $\mathbf{CSSLMet}_u$ but *not* for $\mathbf{SymLMet}_c$ into $\mathbf{CSSLMet}_c$), which is provable without assuming arbitrary choice.

3 Coreflective Subcategories

Example. **Set** is a coreflective subcategory of **Rel**. The function on objects is the powerset constructor $\mathbb{P}$. The coreflection arrow for each set X is the relation $\mathbb{P}(X) \xrightarrow{\ni} X$ given by $\{\langle \mathcal{X}, x\rangle \mid x \in \mathcal{X} \subseteq X\}$. Given a relation $R : Y \to X$, the uniquely induced function from Y to $\mathbb{P}(X)$ is $\lambda y.\,\{x \in X \mid y\ R\ x\}$.

Example. The restriction of $\overline{\langle X, d\rangle}$ to only Cauchy sequences is the coreflection of $\langle X^{\mathbb{N}}, d_\omega\rangle$ for the subcategory **SymLMet** of the category of symmetric Lawvere metric spaces failing to ensure point inequality.

Example. Given a Lawvere metric space $\langle X, d\rangle$, define the distance function $d_{\max}$ on X as follows:

$$d_{\max}(x, x') = \max\{d(x, x'), d(x', x)\}$$

The Lawvere metric space $\langle X, d_{\max}\rangle$ is symmetric, and the identity function is a **SymLMet**-coreflection arrow from $\langle X, d_{\max}\rangle$ to $\langle X, d\rangle$.

Example. Given a symmetric Lawvere metric space $\langle X, d\rangle$ and some subset X' of X satisfying the following property:

$$\forall x \in X.\ \exists! x' \in X'.\ d(x, x') = 0$$

The Lawvere metric space $\langle X', d\rangle$ is symmetric and separated, and the inclusion function for $X' \subseteq X$ is a **SSLMet**-coreflection arrow from $\langle X', d\rangle$ to $\langle X, d\rangle$.