Universal and Existential Quantification

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April 26, 2018

Remark. As with indexed categories, it is best to first develop intuition using propositions. In logics with universal quantification, one often expresses the rules for universal quantification as follows:

$$\frac{\Gamma, x : \tau \mid \Phi \vdash \psi}{\Gamma \mid \Phi \vdash \forall x : \tau.\, \psi} \text{ where } x \text{ is not free in } \Phi \qquad\qquad \frac{\Gamma \mid \Phi \vdash \forall x : \tau.\, \psi}{\Gamma \mid \Phi \vdash \psi[x \mapsto e]} \text{ where } \Gamma \vdash e : \tau$$

First off, Γs are type contexts, meaning they are lists of "$x : \tau$"s indicating what type each available variable has; and Φs are propositional contexts, meaning they are lists of propositions that are assumed to be true. The judgement $\Gamma \mid \Phi \vdash \psi$ conceptually says: with variable types as specified by Γ, the conclusion ψ can be proven under the assumptions Φ—and implicitly ψ and Φ are well-typed propositions in context Γ. The above are inference rules, meaning that the conclusion of the rule, i.e. the judgement below the line, holds if the assumptions of the rule, i.e. the judgements above the line, hold.

The left inference rule says that we can prove $\forall x : \tau.\, \psi$ holds in some context if we can know that ψ holds with no assumptions specifically about x (which is the intuitive meaning of the side condition). The right inference rule says that, if we can prove $\forall x : \tau.\, \psi$ holds in some context, then we know that, for any expression e that can be given type τ in context Γ (which is the intuitive meaning of the side condition), the substituted proposition $\psi[x \mapsto e]$ holds.

Look at both rules and notice that, in both cases, Φ is a well-typed propositional context in context Γ, whereas ψ is a well-typed proposition in context $\Gamma, x : \tau$. So these two rules are in a sense describing a relation between propositional contexts Φ and propositions ψ typed in slightly different contexts, namely the proposition ψ is typed in the context with an additional variable x of type τ.

Given a propositional context Φ typed in context Γ, we can view it as a propositional context typed in context $\Gamma, x : \tau$. This is because there is an assignment of variables $\pi : (\Gamma, x : \tau) \to \Gamma$ that simply assigns each variable in Γ to itself as an expression and forgets the variable x, and so we given Φ in context Γ we can substitute with this assignment to get $\Phi[\pi]$ in context $\Gamma, x : \tau$. But because π is essentially invisible *syntactically*, the *syntactic* result of $\Phi[\pi]$ is simply Φ.

On the other hand, given a proposition ψ typed in context $\Gamma, x : \tau$, the universally-quantified proposition $\forall x : \tau.\, \psi$ is typed in context Γ *without* x. Thus, whereas substituting with π maps propositional contexts typed in context Γ to propositional contexts typed in context $\Gamma, x : \tau$, universal quantification maps propositions typed in context $\Gamma, x : \tau$ to propositions typed in Γ. That is, the two mappings are in opposite directions. And in fact, the inference rules make these mappings adjunctions, giving a transposition between the top and bottom implications below:

$$\frac{\Phi[\pi] \quad \vdash \quad \psi}{\Phi \quad \vdash \quad \forall x : \tau.\, \psi} \text{ where } \Gamma \vdash \Phi \text{ and } \Gamma, x : \tau \vdash \psi$$

Definition (Simple Products)**.** Suppose $\mathbf{I}$ has binary products. For any two objects I and J of $\mathbf{I}$, let the morphisms $\pi_{I,J} : I \times J \to I$ denote the appropriate projection. An $\mathbf{I}$-indexed category $\mathbf{C} : \mathbf{I}^{\mathrm{op}} \to \mathbf{Cat}$ has (strict) simple products if the functor $\mathbf{C}(\pi_{I,J}) : \mathbf{C}(I) \to \mathbf{C}(I \times J)$ has a right adjoint $\prod_{I,J} : \mathbf{C}(I \times J) \to \mathbf{C}(I)$, and furthermore this construction is natural with respect to I, meaning for every morphism $a : I \to I'$ the following diagram commutes:

$$\begin{tikzcd}
\mathbf{C}(I' \times J) \arrow[r, "\prod_{I',J}"] \arrow[d, "\mathbf{C}(a \times J)"'] & \mathbf{C}(I') \arrow[d, "\mathbf{C}(a)"] \\
\mathbf{C}(I \times J) \arrow[r, "\prod_{I,J}"'] & \mathbf{C}(I)
\end{tikzcd}$$

Remark. In the above, I represents Γ and J represents $x : \tau$. The functor $\mathbf{C}(\pi_{I,J})$ representings weakening, and the functor $\prod_{I,J}$ represents universal quantification. The adjunction captures the transposition described above. The naturality condition simply captures the expectation that substitution of the variables in Γ goes through the universal quantification, i.e. the substitution $(\forall x : \tau.\, \psi)[a]$ reduces to $\forall x : \tau.\, (\psi[a])$.

Remark. On the flipside, the inference rules for existential quantification are as follows:

$$\frac{\Gamma, x : \tau \mid \Phi, \psi \vdash \phi}{\Gamma \mid \Phi, \exists x : \tau.\, \psi \vdash \phi} \text{ where } x \text{ is not free in } \Phi \text{ or } \phi \qquad \frac{\Gamma \mid \Phi \vdash \psi[x \mapsto e]}{\Gamma \mid \Phi \vdash \exists x : \tau.\, \psi} \text{ where } \Gamma \vdash e : \tau$$

Note that these are essentially dual to the rules for universal quantification. In fact, they give a transposition between the top and bottom implications below:

$$\frac{\Phi[\pi], \psi \quad \vdash \quad \phi[\pi]}{\Phi, \exists x : \tau.\, \psi \quad \vdash \quad \phi} \text{ where } \Gamma \vdash \Phi \text{ and } \Gamma \vdash \phi \text{ and } \Gamma, x : \tau \vdash \psi$$

One subtle complication here, though, is that there are other propositions on the left-hand side besides ψ. That is, there is some additional context given by Φ. We will return to this subtlety in a bit, but first we consider the situation when there is always exactly one proposition in the list of assumptions, in particular making Φ in this discussion empty.

Definition (Simple Coproducts)**.** Suppose $\mathbf{I}$ has binary products. For any two objects I and J of $\mathbf{I}$, let the morphisms $\pi_{I,J} : I \times J \to I$ denote the appropriate projection. An $\mathbf{I}$-indexed category $\mathbf{C} : \mathbf{I}^{\mathrm{op}} \to \mathbf{Cat}$ has (strict) simple coproducts if the functor $\mathbf{C}(\pi_{I,J}) : \mathbf{C}(I) \to \mathbf{C}(I \times J)$ has a left adjoint $\coprod_{I,J} : \mathbf{C}(I \times J) \to \mathbf{C}(I)$, and furthermore this construction is natural with respect to I, meaning for every morphism $a : I \to I'$ the following diagram commutes:

$$\begin{tikzcd}
\mathbf{C}(I' \times J) \arrow[r, "\coprod_{I',J}"] \arrow[d, "\mathbf{C}(a \times J)"'] & \mathbf{C}(I') \arrow[d, "\mathbf{C}(a)"] \\
\mathbf{C}(I \times J) \arrow[r, "\coprod_{I,J}"'] & \mathbf{C}(I)
\end{tikzcd}$$

Remark. In the above, I represents Γ and J represents $x : \tau$. The functor $\mathbf{C}(\pi_{I,J})$ representings weakening, and the functor $\coprod_{I,J}$ represents existential quantification. The adjunction captures the transposition described above (for empty Φ). The naturality condition simply captures the expectation that substitution of the variables in Γ goes through the existential quantification, i.e. the substitution $(\exists x : \tau.\, \psi)[a]$ reduces to $\exists x : \tau.\, (\psi[a])$.

Definition (Fibred Finite Products)**.** An indexed category $\mathbf{C} : \mathbf{I}^{\mathrm{op}} \to \mathbf{Cat}$ has fibred finite products if, for every object I in $\mathbf{I}$ the category $\mathbf{C}(I)$ has finite products, and for every morphism $a : I \to J$ the functor $\mathbf{C}(a)$ preserves finite products.

Remark. In non-linear logics, we can represent a propositional context Φ by the conjunction of all its elements, i.e. $\phi_1 \wedge \cdots \wedge \phi_n$. Logical conjunction is simply a categorical product within a given context on fibre. If something implies ψ_1 and also implies ψ_2, then it implies $\psi_1 \wedge \psi_2$, and in particular $\psi_1 \wedge \psi_2$ implies both ψ_1 and ψ_2. Thus every fibre of a logic with conjunction (and $\top$) has finite products. Furthermore, since substitutions $(\psi_1 \wedge \psi_2)[a]$ reduce to $(\psi_1[a]) \wedge (\psi_2[a])$, the substitution functors preserve finite products.

Definition (Simple Coproducts satisfying Frobenius)**.** Given an indexed category $\mathbf{C} : \mathbf{I}^{\mathrm{op}} \to \mathbf{Cat}$ with fibred finite products and simple coproducts, for every pair of objects I and J of $\mathbf{I}$ and every object C of $\mathbf{C}(I)$ and object D of $\mathbf{C}(I \times J)$ there is a particularly important morphism that is induced by the structure of fibred finite products and simple coproducts, which is a morphism of the category $\mathbf{C}(I)$ with the following signature:

$$\coprod_{I,J} \mathbf{C}(a)(C) \times D \to C \times \coprod_{I \times J} D$$

$\mathbf{C}$ is said to have simple coproducts satisfying Frobenius if induced $\mathbf{C}(I)$-morphism with the signature above has an inverse.

Remark. If one were to formulate propositional contexts using indexed *multi*categories rather than indexed categories, then simple coproducts satisfying Frobenius correspond to strong simple coproducts, i.e. simple coproducts that can tolerate additional (propositional) context.