Multicategories

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Definition (Multicategory). A multicategory is comprised of the following components:

**Objects** A set Ob of “objects”

**Multimorphisms**: For every list of objects $\vec{C} \in \text{Ob}$ and object $C' \in \text{Ob}$, a set $\text{Hom}(\vec{C},C')$ of “multimorphisms from $\vec{C}$ to $C'$”. Multimorphisms are often illustrated as shown below:

$$C_1 \xrightarrow{m} C'$$

A multimorphism $m : [C_1, C_2, C_3] \rightarrow C'$

$$\vec{C}_1 \xrightarrow{m} \vec{C}'$$

A multimorphism $m : \vec{C}_1 + [C_2] + \vec{C}_3 \rightarrow C'$

**Identities** For all objects $C \in \text{Ob}$, a multimorphism $id_C : [C] \rightarrow C$

**Composition** For all (possibly empty) lists of multimorphisms $m_1 : \vec{C}_1 \rightarrow C_1$, $\ldots$, $m_n : \vec{C}_n \rightarrow C'_n$ and multimorphisms $m' : [C'_1, \ldots, C'_n] \rightarrow C''$, a multimorphism $\Delta_{m_1, \ldots, m_n} : \vec{C}_1 + \ldots + \vec{C}_n \rightarrow C''$

**Identity** For all multimorphisms $m : [C_1, \ldots, C_n] \rightarrow C'$, the equalities $\Delta_{id_{C_1}, \ldots, id_{C_n}} m = m = \Delta_m id_{C''}$ hold

**Associativity** For all lists of lists of multimorphisms $\vec{m}_1, \ldots, \vec{m}_n$, lists of multimorphisms $m_1', \ldots, m_n'$, and multimorphisms $m''$, the equality $\Delta_{m_1', \ldots, m_n'} m' = \Delta_{\vec{m}_1, \ldots, \vec{m}_n} m''$ holds when well typed

Remark. Just like the identity and associativity requirements of categories makes composing paths of morphisms unambiguous, the identity and associativity requirements of multicategories makes composing trees of multimorphisms unambiguous. Thus compositions of multimorphisms in multicategories are often described by illustration, as below:

![Diagram](image)

Definition (Multigraph). A multigraph is a set $V$ of “vertices” and a set $E$ of “edges” along with “source” and “target” functions $s : E \rightarrow LV$ and $t : E \rightarrow V$. The category Multigraph is the comma category $\text{Set} \downarrow ((L \cdot \times \cdot)$.

Definition (Tree). Given sets $L$ and $B$, we inductively define the set $\mathbb{T}_L B$ as having two constructors $\text{leaf}_B : L \rightarrow \mathbb{T}_L B$ and $\text{branch}_L : L(\mathbb{T}_L B) \times B \rightarrow \mathbb{T}_L B$.

Definition (Multipath). Given a multigraph $\langle V,E,s,t \rangle$, we define functions $s_T : \mathbb{T}_V E \rightarrow LV$ and $t_T : \mathbb{T}_V E \rightarrow V$ as follows:

$$s_T(\text{leaf}_E(v)) = [v] \quad t_T(\text{leaf}_E(v)) = v$$

$$s_T(\text{branch}_V([t_1, \ldots, t_n], e)) = s_T(t_1) + \ldots + s_T(t_n) \quad t_T(\text{branch}_V(t, e)) = t(e)$$

A multipath of $\langle V,E,s,t \rangle$ is a tree $t \in \mathbb{T}_V E$ satisfying colored($t$) as defined below:

$$\text{colored}(\text{leaf}_E(v)) = \xi$$

$$\text{colored}(\text{branch}_V([t_1, \ldots, t_n], e)) = \text{colored}(t_1) \land \cdots \land \text{colored}(t_n) \land [t_T(t_1), \ldots, t_T(t_n)] = s(e)$$

This definition has an obvious extension to a monad $\text{Multipath} : \text{Multigraph} \rightarrow \text{Multigraph}$.

Remark. Multicategories are the monad algebras of Multipath.
Example. The multicategory Set has sets as its objects, and its multimorphisms from $[A_1, \ldots, A_n]$ to $B$ are $n$-ary functions $f : A_1 \times \cdots \times A_n \to B$.

Example. Prost has preordered sets as its objects, and its multimorphisms from $[(A_1, \leq_1), \ldots, (A_n, \leq_n)]$ to $(B, \leq)$ are $n$-ary functions $f : A_1 \times \cdots \times A_n \to B$ satisfying the following:

\[
\forall a_1, a'_1 \in A_1, \ldots, a_n, a'_n \in A_n. a_1 \leq_1 \cdots \leq_n a_n \leq a'_n \implies f(a_1, \ldots, a_n) \leq f(a'_1, \ldots, a'_n)
\]

Example. Rel(2) has sets with binary relations as its objects, and its multimorphisms from $[(A_1, R_1), \ldots, (A_n, R_n)]$ to $(B, S)$ are $n$-ary functions $f : A_1 \times \cdots \times A_n \to B$ satisfying the following:

\[
\forall a_1 \in A_1, \ldots, a_n \in A_n. \forall i \in \{1, \ldots, n\}, a'_i \in A_i, R_i a'_i \implies f(a_1, \ldots, a_i, \ldots, a_n) S f(a_1, \ldots, a'_i, \ldots, a_n)
\]

Example. Cat has categories as its objects, and its multimorphisms from $[[C_1, \ldots, C_n]]$ to $D$ are $n$-ary functors $F$, meaning $F$ is a mapping of $n$-tuples of $C$ objects to $D$ objects and $n$-tuples of $C$ morphisms to $D$ morphisms (with obvious qualifications) such that:

\[
F(id_{C_1}, \ldots, id_{C_n}) = id_{F(C_1, \ldots, C_n)} \quad F(f_1; g_1, \ldots, f_n; g_n) = F(f_1, \ldots, f_n) ; F(g_1, \ldots, g_n)
\]

Example. Given a grammar of types and a grammar of typed expressions with an appropriate notion of typed free variables and substitution, there is a corresponding multicategory whose objects are types $\tau$, whose morphisms from $[\tau_1, \ldots, \tau_n]$ to $\tau'$ are expressions of type $\tau'$ with free variables of types $\tau_1, \ldots, \tau_n$, whose identities are given by variables, and whose composition is given by substitution.

Definition (Thin). A multicategory is thin if all multimorphisms with the same domain and codomain are equal.

Definition (Multipreorder). Just like a thin category corresponds to a preorder on its set of objects, a thin multicategory corresponds to a multipreorder on its set of objects. For example, the previous example could alternatively be specified by stating that $[m_1, \ldots, m_n] \leq m$ is defined as $m_1 \ast \cdots \ast m_n = m$. More explicitly, a multipreorder on a set $A$ is a subset $\leq$ of $LA \times A$ satisfying the following reflexivity and transitivity properties:

\[
\forall a \in A. \ [a] \leq a
\]

\[
\forall a_1, \ldots, a_n \in LA. a'_1, \ldots, a'_n, a'' \in A. \ a_1 \leq a'_1 \cdots \leq a'_n \leq a'' \implies a_1 + \cdots + a_n \leq a''
\]

Note that an alternative definition for a multipreorder is a lax monad algebra of the monad $L$ on Rel.

Example. Given a preordered monoid $(M, e, \ast, \leq)$, there is a corresponding thin multicategory whose objects are the elements of $M$ and which has a (unique) morphism from $[m_1, \ldots, m_n]$ to $m$ if and only if $m_1 \ast \cdots \ast m_n \leq m$.

Definition (Multiorder). A multiorder on a set $A$ is a multipreorder on $A$ with the following property:

\[
\forall a, a' \in A. \ [a] \leq a' \implies a = a'
\]

Definition (Operad). An operad is a multicategory with exactly one object. As such, one often describes the domain of a multimorphism of an operad simply by its arity.

Example. The terminal multicategory $1$ is the thin operad with exactly one morphism of each arity. In other words, it is the total multipreorder on the

Definition (Multifunctor). A multifunctor $F$ from a multicategory $C$ to a multicategory $D$ is a mapping of $C$ objects to $D$ objects and $C$ multimorphisms to $D$ multimorphisms (with obvious qualifications) such that:

\[
F(id_C) = id_{F(C)} \quad F \left( \bigsqcup_{f_1, \ldots, f_n} g \right) = \bigsqcup_{F(f_1), \ldots, F(f_n)} F(g)
\]

Just as multicategories can equivalently be defined as monad algebras of Multipath, multifunctors can equivalently be defined as morphisms of monad algebras of Multipath.

Example. Multicat has multicategories as its objects, and its multimorphisms from $[[C_1, \ldots, C_n]]$ to $D$ are $n$-ary multifunctors $F$, meaning $F$ is a mapping of $n$-tuples of $C$ objects to $D$ objects and $n$-tuples of $C$ multimorphisms to $D$ multimorphisms (with obvious qualifications) such that:

\[
F(id_{C_1}, \ldots, id_{C_n}) = id_{F(C_1, \ldots, C_n)} \quad F \left( \bigsqcup_{f_1^1, \ldots, f_1^n} g_1, \ldots, \bigsqcup_{f_n^1, \ldots, f_n^m} g_n \right) = \bigsqcup_{F(f_1^1), \ldots, F(f_n^m)} F(g_1, \ldots, g_n)
\]
**Definition** (Internal Monoid of a Multicategory). An internal monoid of a multicategory $C$ is a multifunctor from the terminal multicategory $1$ to $C$. That is, an internal monoid of $C$ is an object $C$ of $C$ along with a collection of multimorphisms $\{ *_n : [C, \ldots, C] \to C \}_{n \in \mathbb{N}}$ such that all compositions of the multimorphisms in $\{ *_n \}_{n \in \mathbb{N}}$ with the same codomain are equal. One can prove that an internal monoid could equivalently be defined as an object $C$ with two multimorphisms $*_0 : [ ] \to C$ and $*_2 : [C, C] \to C$ satisfying $\Delta_{*_0, *_2} = *_2 = \Delta_{*_0, *_2}$ and $\Delta_{*_2, *_2} = \Delta_{*_2, *_2}$.

**Example.** An internal monoid of $\text{Set}$ is simply a monoid. An internal monoid of $\text{Prost}$ is a preordered monoid, meaning $\forall a_1, a_1', a_2, a_2' \in A. a_1 \leq a_1' \land a_2 \leq a_2' \implies a_1 * a_2 \leq a_1' * a_2'$.

**Definition** (Path-Multicategory). There is a common pattern of multicategories whose objects have left and right “anchors”. Furthermore, often wants consider multifunctors that preserve these anchors. As such, its best to formalize this pattern as a Path-multicategory, more commonly known as an fc-multicategory (where fc stands for “free category”) or a virtual double category. A Path-multicategory is comprised of the following components:

0-cells A set $\text{Ob}$ of “0-cells” or “objects”

Vertical 1-cells For every pair of 0-cells $C$ and $D$, a set $\text{Vom}(C, D)$ of “vertical 1-cells from $C$ to $D$”

Horizontal 1-cells For every pair of 0-cells $C$ and $D$, a set $\text{Hom}(C, D)$ of “horizontal 1-cells from $C$ to $D$”

2-cells For every path of horizontal 1-cells $C_0 \xrightarrow{h_1} C_1 \ldots C_{n-1} \xrightarrow{h_n} C_n$, every horizontal 1-cell $D \xrightarrow{h'} D'$, and every vertical 1-cells $v : C_0 \to D$ and $v' : C_n \to D'$, a set $\text{Face}_{v,v'}([h_0, \ldots, h_n], h')$ of “2-cells from $[h_1, \ldots, h_n]$ to $h'$ along $v$ and $v'$”. 2-cells are often illustrated as shown below:

$$
\begin{array}{ccc}
D & \xrightarrow{\alpha} & D' \\
\downarrow v & & \downarrow v' \\
C_0 \xrightarrow{h_1} C_1 \ldots \xrightarrow{h_n} C_n & \xrightarrow{\beta} & C_n \xrightarrow{h_n} C_n
\end{array}
$$

Vertical 1-identities For every 0-cell $C$, a vertical 1-cell $\text{id}_C : C \to C$

Vertical 1-composition For every vertical 1-cells $C \xrightarrow{v} D \xrightarrow{v'} E$, a vertical 1-cell $C \xrightarrow{v; v'} E$

2-identities For every horizontal 1-cell $h : C \to C'$, a 2-cell $\text{id}_C \xrightarrow{\alpha} \text{id}_C$:

$$
\begin{array}{ccc}
C & \xrightarrow{\alpha} & C' \\
\downarrow h & & \downarrow h' \\
C & \xrightarrow{\beta} & C'
\end{array}
$$

2-composition For every “tree” of 2-cells

$$
\begin{array}{ccc}
E & \xrightarrow{\alpha_1 \ldots \alpha_n \beta} & E' \\
\downarrow \Delta_{\alpha_1 \ldots \alpha_n \beta} & & \downarrow \Delta_{\alpha_1 \ldots \alpha_n \beta} \\
C_0 \xrightarrow{h_1} C_1 \ldots \xrightarrow{h_n} C_n & \xrightarrow{\alpha_1 \ldots \alpha_n \beta} & C_n
\end{array}
$$

1-Identity $\forall v : C \to D. \text{id}_D ; v = v = v ; \text{id}_D$

1-Associativity $\forall C \xrightarrow{v} D \xrightarrow{v'} E \xrightarrow{v''} F. (v ; v') ; v'' = v ; (v' ; v'')$

2-Identity $\forall \alpha : [h_1, \ldots, h_n] \Rightarrow h'. \Delta_{\text{id}_{h_1}, \ldots, \text{id}_{h_n}} \alpha = \alpha = \Delta_{\alpha} \text{id}_{h'}$

2-Associativity $\forall \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma. \Delta_{\text{id}_{h_1}, \ldots, \text{id}_{h_n}} \alpha = \alpha = \Delta_{\text{id}_{h_1}, \ldots, \text{id}_{h_n}} \gamma$

\[ \Delta_{\text{id}_{h_1}, \ldots, \text{id}_{h_n}} \alpha \overset{\gamma}{\Rightarrow} \Delta_{\text{id}_{h_1}, \ldots, \text{id}_{h_n}} \gamma \]

\[ \Delta_{\text{id}_{h_1}, \ldots, \text{id}_{h_n}} = \Delta_{\text{id}_{h_1}, \ldots, \text{id}_{h_n}} \gamma \]

\[ \Delta_{\text{id}_{h_1}, \ldots, \text{id}_{h_n}} = \Delta_{\text{id}_{h_1}, \ldots, \text{id}_{h_n}} \gamma \]
Example. BinRel has sets as its objects, functions as its vertical morphisms, and binary relations as its horizontal morphisms. It has a unique 2-cell from \([R_1 \subseteq A_0 \times A_1, \ldots, R_n \subseteq A_{n-1} \times A_n]\) to \(S \subseteq B \times B'\) along \(f : A_0 \to B\) and \(f' : A_n \to B'\) if and only if the following holds:

\[
\forall a_0 : A_0, \ldots, a_n : A_n. \ a_0 R_1 a_1 \land \cdots \land a_{n-1} R_n a_n \implies f(a_0) S f'(a_n)
\]

Example. SplitGraph has sets “of vertices” as its objects and “vertex mapping” functions as its vertical morphisms. A horizontal morphism from \(V \to V'\) is a set \(E\) of “edges” along with a “source” function \(s : E \to V\) and a “target” function \(t : E \to V'\). Given a chain of horizontal morphisms \([V_0 \overset{r_1}{\to} E_1 \overset{t_1}{\to} V_1, \ldots, V_{n-1} \overset{r_{n-1}}{\to} E_n \overset{t_n}{\to} V_n]\), a “path” of this chain is a sequence \([v_0 \in V_0, e_1 \in E_1, v_1 \in V_1, \ldots, v_{n-1} \in V_{n-1}, e_n \in E_n, v_n \in V_n]\) with the property that \(v_i \in \{1, \ldots, n\}, v_{i-1} = s(e_i) \land t_i(e_i) = v_n\). A 2-cell from \([V_0 \overset{r_1}{\to} E_1 \overset{t_1}{\to} V_1, \ldots, V_{n-1} \overset{r_{n-1}}{\to} E_n \overset{t_n}{\to} V_n]\) to \(V' \overset{s'}{\to} E' \overset{t'}{\to} V''\) along \(f_V : V_0 \to V'\) and \(f'_V : V_n \to V''\) is a “path mapping” function \(f_E\) mapping each path \([v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n]\) of the domain to an element \(e'\) of \(E'\) such that \(f_E(v_0) = s'(e')\) and \(t'(e') = f'_V(v_n)\).

Example. Given a 2-category, there is a corresponding \(\text{Path}\)-multicategory with the same 0-cells, whose horizontal and vertical 1-cells are both the 1-cells of the 2-category, and whose 2-cells from \([h_1, \ldots, h_n]\) to \(h'\) along \(v\) and \(v'\) are the 2-cells of the 2-category from \(h_1; \ldots; h_n; v'\) to \(v; h'\).

Definition (Path-Multifunctor). A \(\text{Path}\)-multifunctor \(F\) from a \(\text{Path}\)-multicategory \(C\) to a \(\text{Path}\)-multicategory \(D\) is a mapping of \(C\) objects to \(D\) objects, \(C\) vertical multimorphisms to \(D\) vertical morphisms, \(C\) horizontal multimorphisms to \(D\) horizontal morphisms, and \(C\) 2-cells to \(D\) 2-cells (with obvious qualifications) such that:

\[
F(id_C) = id_{F(C)} \quad F(v; v') = F(v); F(v') \quad F(id_h) = id_{F(h)} \quad F(\Delta_{\alpha_1, \ldots, \alpha_n}) = \Delta_{\Delta_{\alpha_1}, \ldots, \Delta_{\alpha_n}} F(\beta)
\]

Definition (Internal Monoid of a \(\text{Path}\)-Multicategory). An internal monoid of a \(\text{Path}\)-multicategory \(C\) is a \(\text{Path}\)-multifunctor from the terminal multicategory \(1\) to \(C\). That is, an internal monoid of \(C\) is an object \(C\) of \(C\) along with a horizontal morphism \(m : C \to C\) and a collection of 2-cells \(\{\mu_n \in \text{Face}_{id_C, id_C}([m, \ldots, (i \text{ occurrences of } m)], m)\}_{n \in \mathbb{N}}\) such that all compositions of the 2-cells in \(\{\mu_n\}_{n \in \mathbb{N}}\) with the same codomain are equal. One can prove that an internal monoid could equivalently be defined as an object \(C\) with a horizontal morphism \(m : C \to C\) and two 2-cells \(\mu_0 \in \text{Face}_{id_C, id_C}([], m)\) and \(\mu_2 \in \text{Face}_{id_C, id_C}([m, m], m)\) satisfying \(\Delta_{id_m, \mu_0} \mu_2 = \mu_2 = \Delta_{\mu_0, id_m} \mu_2\) and \(\Delta_{\mu_2, id_m} \mu_2 = \Delta_{id_m, \mu_2 \mu_2}\).

Example. An internal monoid of BinRel is a preordered set. An internal monoid of SplitGraph is a category. An internal monoid of a 2-category is a Path-multicategory is a monad in that 2-category.

Definition (Natural Transformation of \(\text{Multifunctors}\)). Given multifunctors \(F\) and \(G\) from multicategory \(C\) to multicategory \(D\), a natural transformation from \(F\) to \(G\) is a collection of multimorphisms \(\{\alpha_C : [F(C)] \to [G(C)]\}_{C \in C}\) satisfying the property \(\forall f : [C_1, \ldots, C_n] \to [C']\) \(\Delta_{\alpha_{C_1}, \ldots, \alpha_{C_n}} G(f) = \Delta_{F(f)} \alpha_{C'}\).

Definition (Internal Monoid Homomorphism of a Multicategory). Given two internal monoids of a multicategory \(M, M' : 1 \to C\), an internal monoid homomorphism is a natural transformation from \(M\) to \(M'\).

Definition (Natural Transformation of \(\text{Path-Multifunctors}\)). Given \(\text{Path}\)-multifunctors \(F\) and \(G\) from \(C\) to \(D\), a natural transformation from \(F\) to \(G\) consists of a collection of vertical multimorphisms \(\{\nu_C : [F(C)] \to [G(C)]\}_{C \in C}\) and a collection of 2-cells \(\{\theta_h \in \text{Face}_{id_C, id_C}([F(h)], [G(h)])\}_{h \in \text{Hom}^C(C, C')}\) satisfying the following properties:

\[
\forall v : C \to C'. \ \nu_C ; G(v) = F(v); \alpha_{C'} \quad \text{and} \quad \forall \alpha : [h_1, \ldots, h_n] \to h'. \ \Delta_{\theta_{h_1}, \ldots, \theta_{h_n}} G(\alpha) = \Delta_{\theta_{h'}, \ldots, \theta_{h'}} F(\alpha)
\]

Definition (Modification of \(\text{Natural Transformations of \(\text{Path-Multifunctors}\)}\)). Given \(\text{Path}\)-multifunctors \(F\) and \(G\) from \(C\) to \(D\), a modification from \(\langle \nu, \theta \rangle\) and \(\langle \nu', \theta' \rangle\) from \(\text{Path-multifunctor} F\) to \(\text{Path-multifunctor} G\) from \(C\) to \(D\), a modification from \(\langle \nu, \theta \rangle\) and \(\langle \nu', \theta' \rangle\) is a collection of 2-cells \(\{\mu_h \in \text{Face}_{id_C, id_C}([F(h)], [G(h)])\}_{h \in \text{Hom}^C(C, C')}\) satisfying the following property:

\[
\forall \alpha : [C_0 \overset{h_1}{\to} C_1 \ldots \overset{h_{n-1}}{\to} C_n] \to [h'] \quad \Delta_{\theta_{h_1}, \ldots, \theta_{h_n}} G(\alpha) = \Delta_{\theta_{h'}, \ldots, \theta_{h'}} F(\alpha)
\]
Example. Just as monads correspond to $\text{Path}$-multifunctors from $1$ to a 2-category, oplax monad morphisms between monads correspond to natural transformations between those $\text{Path}$-multifunctors.

Example. Just as preorders correspond to $\text{Path}$-multifunctors from $1$ to $\text{BinRel}$, relation-preserving function between preorders correspond to natural transformations between those $\text{Path}$-multifunctors, and a relation-preserving function is less than another in the usual sense if and only if there exists a (necessarily unique) modification between the corresponding natural transformations.

Example. Just as categories correspond to $\text{Path}$-multifunctors from $1$ to $\text{SplitGraph}$, functors between categories correspond to natural transformations between those $\text{Path}$-multifunctors, and natural transformations between functors correspond to modifications between those natural transformations.

Remark. Given a multicategory, there is a corresponding $\text{Path}$-multicategory called its suspension with exactly one 0-cell, whose only vertical morphism is the identity on that unique 0-cell, whose horizontal morphisms (necessarily from that unique 0-cell to itself) are the objects of the multicategory, and whose 2-cells are the multimorphisms of the multicategory.

In the other direction, given a $\text{Path}$-multicategory there is a corresponding multicategory whose objects are the horizontal 1-cells and whose multimorphisms are the 2-cells, a construction that in a sense forgets the 0-cells and the vertical 1-cells. Suspension and this forgetful construction can be extended to mappings between ($\text{Path}$-)multifunctors, between transformations, and even between modifications, provided one adapts the above definition of modifications for $\text{Path}$-multicategories to plain multicategories in the obvious manner. Putting aside modifications for simplicity, both of these mappings preserve the obvious forms of composition on these structures, meaning they are both 2-functors between the respective 2-categories $\text{Multicat}$ and $\text{Path-Multicat}$. In fact, the suspension 2-functor is right adjoint to the forgetful 2-functor (in the 2-category of 2-categories).