Monoidal Categories

Ross Tate

April 20, 2018

**Definition.** Given a category $\mathbf{C}$, the category $\mathbf{LC}$ is comprised of the following:

**Objects** An object is a list of objects of $\mathbf{C}$.

**Morphisms** A morphism from $[A_1, \ldots, A_n]$ to $[B_1, \ldots, B_n]$ only exists when $m$ equals $n$, in which case it is a list of morphisms $f_1 : A_1 \to B_1, \ldots, f_n : A_n \to B_n$ of $\mathbf{C}$.

**Identities** The identity of $[A_1, \ldots, A_n]$ is $[id_{A_1}, \ldots, id_{A_n}]$.

**Composition** The composition of $[f_1, \ldots, f_n]$ with $[g_1, \ldots, g_n]$ is $[f_1 \circ g_1, \ldots, f_n \circ g_n]$.

**Remark.** The above construction extends to a 2-monad on $\mathbf{Cat}$, with $\eta_{\mathbf{C}}$ being the functor that maps each object and each morphism to the singleton list containing it, and $\mu_{\mathbf{C}}$ being the functor that maps each list of lists of objects and each list of lists of morphisms to its flattening.

**Definition** (Strict Monoidal Category). A strict monoidal category is equivalently defined as a (strict) monad algebra of $\mathbf{L}$ on $\mathbf{Cat}$ or as an internal monoid of the multicategory $\mathbf{Cat}$, i.e. a category $\mathbf{C}$ along with an object $I$ of $\mathbf{C}$ and a binary functor $\otimes : [\mathbf{C}, \mathbf{C}] \to \mathbf{C}$ satisfying identity and associativity. Similarly, a strict monoidal functor is equivalently defined as a (strict) morphism of monad algebras of $\mathbf{L}$ on $\mathbf{Cat}$ or as an internal monoid homomorphism of the multicategory $\mathbf{Cat}$, i.e. a functor $F$ that preserves $I$ and $\otimes$ (strictly). A strict monoidal transformation is defined as a transformation of (strict) morphisms of monad algebras of $\mathbf{L}$ on $\mathbf{Cat}$, or equivalently as a natural transformation $\alpha$ from $F$ to $G$ such that $\alpha_I$ equals $id_I$ and $\alpha_{A \otimes B}$ equals $\alpha_A \otimes \alpha_B$.

**Definition** (Weak Monoidal Category). A weak monoidal category is equivalently defined as a weak monad algebra of $\mathbf{L}$ on $\mathbf{Cat}$ or as the category of unary morphisms of a representable multicategory. Similarly, a weak monoidal functor is equivalently defined as a weak monad algebras of $\mathbf{L}$ on $\mathbf{Cat}$ or as a functor corresponding to a tensor-preserving multifunctor between representable multicategories, i.e. multifunctors $F$ with the property that $FT$ with $Ft$ is a tensor of $FA_1, \ldots, FA_n$ whenever $T$ with $t$ is a tensor of $A_1, \ldots, A_n$. A weak monoidal transformation is defined as a transformation of weak morphisms of weak monad algebras of $\mathbf{L}$ on $\mathbf{Cat}$, or equivalently as a natural transformation $\alpha$ from $F$ to $G$ corresponding to a natural transformation $\Delta_{\alpha_{A_1}, \ldots, \alpha_{A_n}}$ of the natural transformations $\alpha_{A_1} : I \otimes A \to A$ for $A \in \mathbf{C}$, and $\{\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)\}_{A,B,C} \otimes_{\mathbf{C}}$ making the following triangle and pentagon commute:

![Diagram](attachment:image.png)
**Definition (Lax Monoidal Functor).** A lax monoidal functor between strict/weak monoidal categories is equivalently defined as a lax morphism of strict/weak monad algebras of $L$ on $\text{Cat}$ or as a functor corresponding to a (not necessarily tensor-preserving) multifunctor between representable multicategories. A lax monoidal transformation is defined as a transformation of lax morphisms of weak monad algebras of $L$ on $\text{Cat}$, or equivalently as a natural transformation corresponding to a natural transformation of multifunctors.

**Remark.** Yet another equivalent, and particularly common, definition of lax monoidal functor corresponding to the earlier common definition of weak monoidal categories is a functor $F$ along with a morphism $\text{merge} : I' \to FI$ and a natural transformation $\{\text{merge}_{A,B} : FA \otimes' FB \to F(A \otimes B)\}_{A,B \in C}$ such that the following diagrams commute:

\[
\begin{align*}
& I' \otimes' FA \xrightarrow{\text{merge} \otimes' FA} FI \otimes' FA \\
& FA \xleftarrow{F\lambda_A} F(I \otimes A) \\
& FA \xrightarrow{F \rho_A} F(A \otimes I) \\
& (FA \otimes' FB) \otimes' FC \xrightarrow{\alpha_{FA,FB,FC}'} FA \otimes' (FB \otimes' FC) \\
& F((A \otimes B) \otimes C) \xrightarrow{F\alpha_{A,B,C}} F(A \otimes (B \otimes C)) \\
& \text{merge}_{A,B} \otimes' FC \\
& \text{merge}_{A,B,C}
\end{align*}
\]