**Monad Algebras**

Ross Tate

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**Definition** (Monad Algebra). A monad algebra of a Cat-monad \( \langle M : C \to C, \eta, \mu \rangle \), also known as an Eilenberg-Moore algebra, is an object \( A \) of \( C \) along with a morphism \( a : MA \to A \) such that the following both commute:

\[
\begin{array}{ccc}
\eta_A & MA & a \\
\downarrow & \downarrow & \downarrow \\
A & A & id_A \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mu_A & MA & a \\
\downarrow & \downarrow & \downarrow \\
MMA & A & Ma \\
\end{array}
\]

**Example.** The monad algebras for \( L \) coincide with monoids. The monad algebras for \( M \) coincide with commutative monoids. The monad algebras for \( F \) coincide with idempotent (meaning \( \forall x. x * x = x \) commutative monoids. The monad algebras for \( P \) coincide with partial orders with arbitrary joins (by defining \( x \leq x' \) as \( a(\{x, x'\}) = x' \)).

**Definition** (Eilenberg-Moore Category). The Eilenberg-Moore category of a monad \( \langle M : C \to C, \eta, \mu \rangle \), often denoted \( CM \), is the full subcategory of \( \text{Alg}(M) \) comprised of the \( M \)-algebras satisfying the requirements of monad algebras of \( \langle M, \eta, \mu \rangle \). Note that \( CM \) can be viewed as a concrete category over \( C \).

**Example.** The category \( \text{Set}^L \) is concretely isomorphic to \( \text{Mon} \). The category \( \text{Set}^M \) is concretely isomorphic to \( \text{CommMon} \). The category \( \text{Set}^P \) is concretely isomorphic to \( \text{JCPos} \).

**Example.** The category \( \text{Graph}^\text{Path} \) is concretely isomorphic to \( \text{Cat} \).

**Definition** (Premodule of a Monad). Given a monad \( \langle m : C \to C, \eta, \mu \rangle \) of a 2-category \( C \), a premodule, also known as a left module, is a 0-cell \( L \) along with a 1-cell \( \ell : L \to C \) and a 2-cell \( \lambda : \ell ; m \Rightarrow \ell \) satisfying the following equalities:

\[
\begin{array}{ccc}
L & \ell & C \\
\lambda & \downarrow & \downarrow m \\
\lambda & \downarrow & \downarrow \eta \\
\ell & \downarrow & \downarrow \ell \\
L & \ell & L \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
L & \ell & \ell \\
\lambda & \downarrow & \downarrow \ell \\
C & \downarrow & \downarrow C \\
\ell & \downarrow & \downarrow \ell \\
L & \ell & L \\
\end{array}
\]

**Remark.** In terms of string diagrams, the above equalities are formulated as

\[
\begin{array}{ccc}
L & \ell & C \\
\lambda & \downarrow & \downarrow m \\
\lambda & \downarrow & \downarrow \eta \\
\ell & \downarrow & \downarrow \ell \\
L & \ell & L \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
L & \ell & \ell \\
\lambda & \downarrow & \downarrow \ell \\
C & \downarrow & \downarrow C \\
\ell & \downarrow & \downarrow \ell \\
L & \ell & L \\
\end{array}
\]

**Example.** A monad algebra for a Cat-monad is simply a premodule where \( L \) is \( 1 \), \( \ell \) is \( A \), and \( \lambda \) is \( a \).

**Example.** Every monad \( \langle m : C \to C, \eta, \mu \rangle \) is a premodule of itself, with \( L \) as \( C \), \( \ell \) as \( m \), and \( \lambda \) as \( \mu \).

**Example.** For any Cat-monad \( \langle M : C \to C, \eta, \mu \rangle \), the category \( CM \) along with its underlying functor \( U : CM \to C \) and the canonical natural transformation \( \alpha : U ; M \Rightarrow U \) inherited from \( \text{Alg}(M) \) forms a premodule of \( \langle M, \eta, \mu \rangle \). In fact, it is the universal premodule of the monad \( \langle M, \eta, \mu \rangle \).

**Definition** (Eilenberg-Moore Object). An Eilenberg-Moore object \( C^m \) of a given monad \( \langle m : C \to C, \eta, \mu \rangle \) in a 2-category \( C \) is a universal premodule of that monad.

**Remark.** Because every monad is its own premodule, this implies there is a 1-cell \( f : C \to C^m \) (if \( C^m \) exists) such that \( f ; u \) equals \( m \). One can show that these 1-cells always form an adjunction \( f \dashv u \) that gives rise to the monad \( m \).
**Definition (Lax Monad Algebra).** A lax monad algebra of a 2-monad \( \langle M : C \to C, \eta, \mu \rangle \) is a 0-cell \( A \) of the 2-category \( C \) along with a 1-cell \( a : MA \to A \) and 2-cells given below such that the following identity and associativity laws hold:

\[
\begin{align*}
MMA & \xrightarrow{\mu_A} MA \\
Ma & \xrightarrow{\eta} A \\
Ma & \xrightarrow{id_A} MA
\end{align*}
\]

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\begin{align*}
MMA & \xrightarrow{\mu_A} MA \\
Ma & \xrightarrow{\eta} A \\
Ma & \xrightarrow{id_A} MA
\end{align*}
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\begin{align*}
MMA & \xrightarrow{\mu_A} MA \\
Ma & \xrightarrow{\eta} A \\
Ma & \xrightarrow{id_A} MA
\end{align*}
\]

**Definition (Colax Monad Algebra).** The definition of a colax monad algebra is the same as that of a lax monad algebra but with the 2-cells \( \iota \) and \( \gamma \) going in the reverse direction.

**Definition (Weak Monad Algebra).** A weak monad algebra is both a lax and a colax monad algebra in which the opposing \( \iota \)s and opposing \( \gamma \)s are inverses of each other. That is, a weak monad algebra is a lax or colax monad algebra in which \( \iota \) and \( \gamma \) have inverses.

**Definition (Strict Monad Algebra).** A strict monad algebra is both a lax and a colax monad algebra in which both the \( \iota \)s and the \( \gamma \)s are identities. That is, a strict monad algebra is a lax or colax monad algebra in which \( \iota \) and \( \gamma \) are both identities.

**Definition (Lax Morphism of Lax Monad Algebras).** A lax morphism from \( \langle A, a, \iota, \gamma \rangle \) to \( \langle B, b, \iota', \gamma' \rangle \) is a 1-cell \( f : A \to B \) along with a 2-cell \( \alpha : Mf \cdot b \Rightarrow a ; f \) (note the direction) satisfying the following equalities:

\[
\begin{align*}
\begin{align*}
Mf & \xrightarrow{\eta_A} MA \\
Mf & \xrightarrow{id_B} MB \\
f & \xrightarrow{id_A} A
\end{align*}
\end{align*}
\]

\[
\begin{align*}
\begin{align*}
Mf & \xrightarrow{\eta_A} MA \\
Mf & \xrightarrow{id_B} MB \\
f & \xrightarrow{id_A} A
\end{align*}
\end{align*}
\]

\[
\begin{align*}
\begin{align*}
Mf & \xrightarrow{\eta_A} MA \\
Mf & \xrightarrow{id_B} MB \\
f & \xrightarrow{id_A} A
\end{align*}
\end{align*}
\]

\[
\begin{align*}
\begin{align*}
Mf & \xrightarrow{\eta_A} MA \\
Mf & \xrightarrow{id_B} MB \\
f & \xrightarrow{id_A} A
\end{align*}
\end{align*}
\]

**Definition.** A transformation from \( \langle f, \alpha \rangle \) to \( \langle f', \alpha' \rangle \) is a 2-cell \( \theta : f \Rightarrow f' \) such that \( \alpha ; (a * \theta) = (M\theta * b) ; \alpha' \).