Monad Algebras

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Definition (Monad Algebra)**.** A monad algebra of a **Cat**-monad $\langle M : \mathbf{C} \to \mathbf{C}, \eta, \mu \rangle$, also known as an Eilenberg-Moore algebra, is an object A of $\mathbf{C}$ along with a morphism $a : MA \to A$ such that the following both commute:

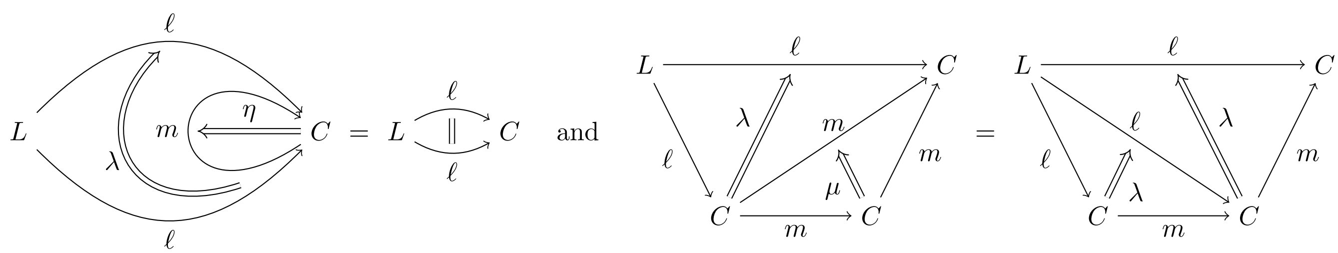
Example. The monad algebras for $\mathbb{L}$ coincide with monoids. The monad algebras for $\mathbb{M}$ coincide with commutative monoids. The monad algebras for $\mathbb{F}$ coincide with idempotent (meaning $\forall x.\ x * x = x$) commutative monoids. The monad algebras for $\mathbb{P}$ coincide with partial orders with arbitrary joins (by defining $x \leq x'$ as $a(\{x, x'\}) = x'$).

Definition (Eilenberg-Moore Category)**.** The Eilenberg-Moore category of a monad $\langle M : \mathbf{C} \to \mathbf{C}, \eta, \mu \rangle$, often denoted $\mathbf{C}^M$, is the full subcategory of $\mathbf{Alg}(M)$ comprised of the M-algebras satisfying the requirements of monad algebras of $\langle M, \eta, \mu \rangle$. Note that $\mathbf{C}^M$ can be viewed as a concrete category over $\mathbf{C}$.

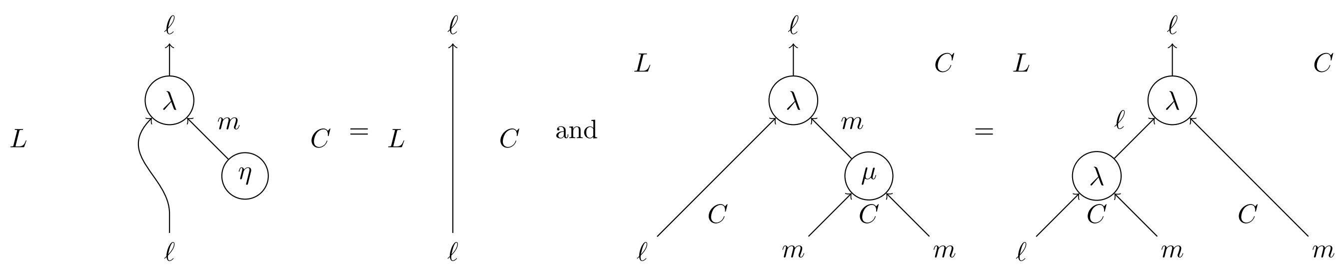
Example. The category $\mathbf{Set}^{\mathbb{L}}$ is concretely isomorphic to **Mon**. The category $\mathbf{Set}^{\mathbb{M}}$ is concretely isomorphic to **CommMon**. The category $\mathbf{Set}^{\mathbb{P}}$ is concretely isomorphic to **JCPos**.

Example. The category $\mathbf{Graph}^{\mathrm{Path}}$ is concretely isomorphic to **Cat**.

Definition (Premodule of a Monad)**.** Given a monad $\langle m : C \to C, \eta, \mu \rangle$ of a 2-category $\mathbf{C}$, a premodule, also known as a left module, is a 0-cell L along with a 1-cell $\ell : L \to C$ and a 2-cell $\lambda : \ell \,;m \Rightarrow \ell$ satisfying the following equalities:



Remark. In terms of string diagrams, the above equalities are formulated as



Example. A monad algebra for a **Cat**-monad is simply a premodule where L is $\mathbf{1}$, ℓ is A, and λ is a.

Example. Every monad $\langle m : C \to C, \eta, \mu \rangle$ is a premodule of itself, with L as C, ℓ as m, and λ as μ.

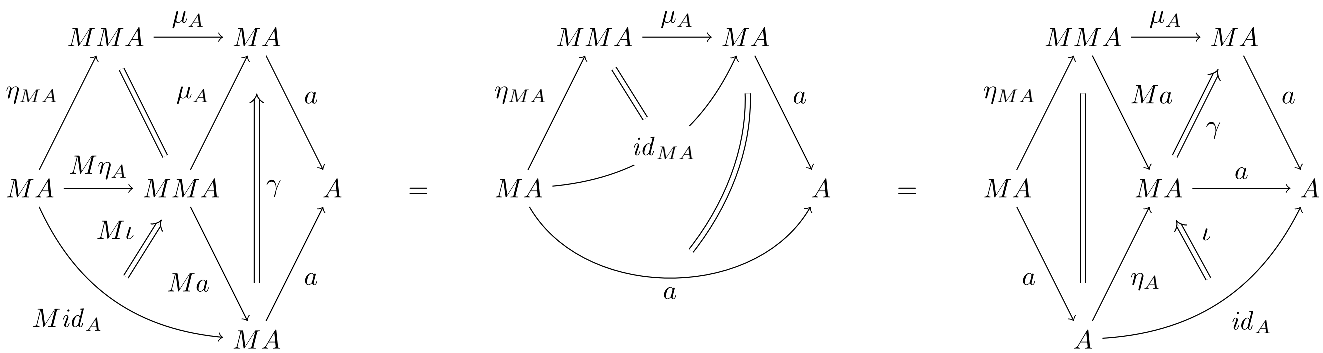
Example. For any **Cat**-monad $\langle M : \mathbf{C} \to \mathbf{C}, \eta, \mu \rangle$, the category $\mathbf{C}^M$ along with its underlying functor $U : \mathbf{C}^M \to \mathbf{C}$ and the canonical natural transformation $\alpha : U \,; M \Rightarrow U$ inherited from $\mathbf{Alg}(M)$ forms a premodule of $\langle M, \eta, \mu \rangle$. In fact, it is the *universal* premodule of the monad $\langle M, \eta, \mu \rangle$.

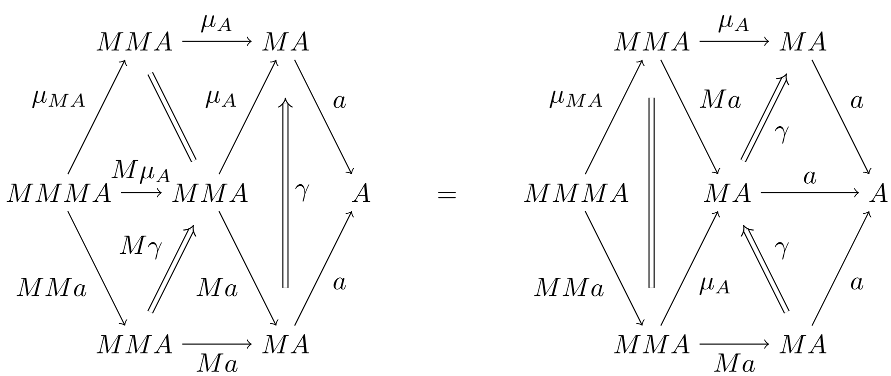
Definition (Eilenberg-Moore Object)**.** An Eilenberg-Moore object C^m of a given monad $\langle m : C \to C, \eta, \mu \rangle$ in a 2-category $\mathbf{C}$ is a universal premodule of that monad.

Remark. Because every monad is its own premodule, this implies there is a 1-cell $f : C \to C^m$ (if C^m exists) such that $f \,; u$ equals m. One can show that these 1-cells always form an adjunction $f \dashv u$ that gives rise to the monad m.

Definition (Lax Monad Algebra)**.** A lax monad algebra of a 2-monad $\langle M : \mathbf{C} \to \mathbf{C}, \eta, \mu \rangle$ is a 0-cell A of the 2-category $\mathbf{C}$ along with a 1-cell $a : MA \to A$ and 2-cells given below

such that the following identity and associativity laws hold:



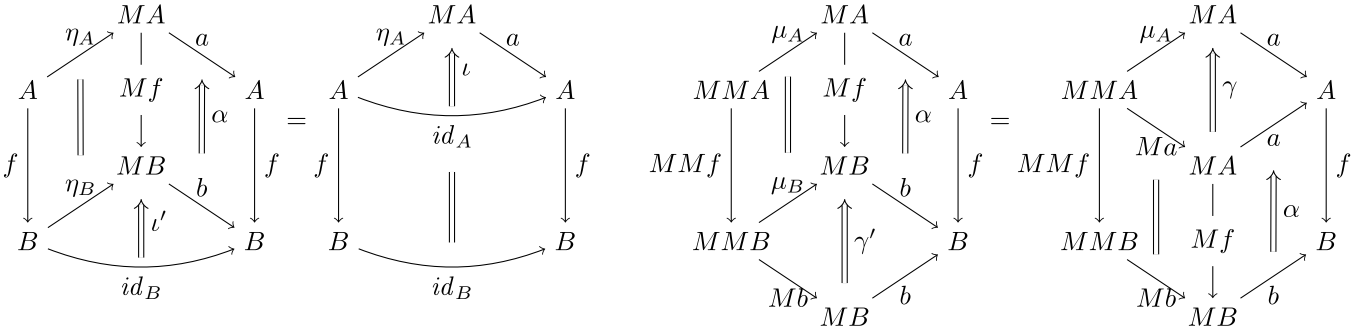


Definition (Colax Monad Algebra)**.** The definition of a colax monad algebra is the same as that of a lax monad algebra but with the 2-cells ι and γ going in the reverse direction.

Definition (Weak Monad Algebra)**.** A weak monad algebra is both a lax and a colax monad algebra in which the opposing ιs and opposing γs are inverses of each other. That is, a weak monad algebra is a lax or colax monad algebra in which ι and γ have inverses.

Definition (Strict Monad Algebra)**.** A strict monad algebra is both a lax and a colax monad algebra in which both the ιs and the γs are identities. That is, a strict monad algebra is a lax or colax monad algebra in which ι and γ are both identities.

Definition (Lax Morphism of Lax Monad Algebras)**.** A lax morphism from $\langle A, a, \iota, \gamma \rangle$ to $\langle B, b, \iota', \gamma' \rangle$ is a 1-cell $f : A \to B$ along with a 2-cell $\alpha : Mf \,; b \Rightarrow a \,; f$ (note the direction) satisfying the following equalities:



Definition. A transformation from $\langle f, \alpha \rangle$ to $\langle f', \alpha' \rangle$ is a 2-cell $\theta : f \Rightarrow f'$ such that $\alpha \,;(a * \theta) = (M\theta * b) \,; \alpha'$.