1 Metric Spaces

Definition. $\mathbb{R}^{\geq \infty}$ is the set of non-negative real numbers along with the “value” $\infty$, i.e. $\mathbb{R}^\geq \cup \{\infty\}$. Addition, multiplication, and inequality are defined on $\infty$ as one would expect.

Definition. A Lawvere metric space, also known as a pseudoquasimetric space, is a set $X$ and a “distance” function $d : X \times X \to \mathbb{R}^{\geq \infty}$ satisfying the following properties:

- **Point Inequality** $\forall x \in X. \ 0 \geq d(x, x)$
- **Triangle Inequality** $\forall x, y, z \in X. d(x, y) + d(y, z) \geq d(x, z)$

Definition. A metric space is a Lawvere metric space $\langle X, d \rangle$ satisfying the following additional properties:

- **Finiteness** $\forall x, y \in X. d(x, y) \neq \infty$
- **Symmetry** $\forall x, y \in X. d(x, y) = d(y, x)$
- **Separation** $\forall x, y \in X. d(x, y) = 0 \implies x = y$

Example. Given a “dimension” $D$ (which in general is just some set) and a “power” $p \in \mathbb{R}^>$, define $d_p : \mathbb{R}^D \times \mathbb{R}^D \to \mathbb{R}^{\geq \infty}$ as follows:

$$d_p(\vec{x}, \vec{y}) = \sqrt[p]{\sum_{i \in D} |y_i - x_i|^p}$$

The pair $\langle \mathbb{R}^D, d_p \rangle$ is a symmetric and separable Lawvere metric space that is also finite whenever $D$ is finite. That is, $\langle \mathbb{R}^D, d_p \rangle$ is a metric space whenever $D$ is finite. When $p$ is 2, this is the Euclidean metric space. When $p$ is 1, this is the Manhattan metric space.

Definition. A metric map (also known as a nonexpansive function or a weak contraction) from a (Lawvere) metric space $\langle X, d_X \rangle$ to a (Lawvere) metric space $\langle Y, d_Y \rangle$ is a function $f : X \to Y$ satisfying the following property:

$$\forall x, x' \in X. \ d_X(x, x') \geq d_Y(f(x), f(x'))$$

Definition. A uniformly continuous function from a (Lawvere) metric space $\langle X, d_X \rangle$ to a (Lawvere) metric space $\langle Y, d_Y \rangle$ is a function $f : X \to Y$ satisfying the following property:

$$\forall \varepsilon \in \mathbb{R}^>. \ \exists \delta \in \mathbb{R}^>. \ \forall x, x' \in X. \ d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \varepsilon$$

Definition. A continuous function from a (Lawvere) metric space $\langle X, d_X \rangle$ to a (Lawvere) metric space $\langle Y, d_Y \rangle$ is a function $f : X \to Y$ satisfying the following property:

$$\forall x \in X. \ \forall \varepsilon \in \mathbb{R}^>. \ \exists \delta \in \mathbb{R}^>. \ \forall x' \in X. \ d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \varepsilon$$

Definition. $\text{(L)Met}$ is the category of (Lawvere) metric spaces and metric maps. $\text{(L)Met}_u$ is the category of (Lawvere) metric spaces and uniformly continuous functions. $\text{(L)Met}_c$ is the category of (Lawvere) metric spaces and continuous functions.
2 Topologies

Definition. A topological space is a set \( X \) along with a “topology” \( \tau \subseteq \mathcal{P}X \) of “open” sets that is closed under finite intersections and arbitrary unions, meaning it satisfies the following properties:

- \( X \in \tau \)
- \( \forall O, O' \in \tau. O \cap O' \in \tau \)
- \( \forall O \subseteq \tau. \bigcup O \in \tau \)

Example. The Sierpiński space \( S \) is the pair \( \langle B, \{\emptyset, \{t\}, B\} \rangle \).

Definition. Given a function \( f : X \to Y \) and a subset \( Y \) of \( Y \), the preimage of \( Y \) under \( f \), which is denoted \( f^{-1}(Y) \), is the set \( \{x \in X \mid f(x) \in Y\} \).

Definition. A continuous function from a topological space \( \langle X, \tau_X \rangle \) to a topological space \( \langle Y, \tau_Y \rangle \) is a function \( f : X \to Y \) whose preimage function preserves openness, meaning the following property holds:

\[ \forall O \in \tau_Y. f^{-1}(O) \in \tau_X \]

Definition. \textbf{Top} is the category of topological spaces and continuous functions.

3 Functors

Definition. Every metric map is a uniformly continuous function, and every uniformly continuous function is a continuous function, so there are “obvious” inclusion functors \( (\text{L})\text{Met} \hookrightarrow (\text{L})\text{Met}_u \hookrightarrow (\text{L})\text{Met}_c \).

Definition. \( \mathbb{R}^{\geq \infty} \) is the set of positive real numbers along with the “value” \( \infty \), i.e. \( \mathbb{R}^{\geq} \cup \{\infty\} \).

Definition. For a (Lawvere) metric space \( \langle X, d \rangle \), the open ball of radius \( r \in \mathbb{R}^{\geq \infty} \) around an element \( x \) of \( X \), denoted \( B_r(x) \), is the set \( \{x' \in X \mid d(x, x') < r\} \).

Definition. Given a (Lawvere) metric space \( \langle X, d \rangle \), the corresponding topology \( \tau_d \) is the set of all subsets \( O \subseteq X \) satisfying the following property:

\[ \forall x \in O. \exists r \in \mathbb{R}^{\geq \infty}. B_r(x) \subseteq O \]

Definition. As an abuse of notation, \( \tau : (\text{L})\text{Met}_{(u/c)} \to \text{Top} \) is the functor mapping a (Lawvere) metric space \( \langle X, d \rangle \) to the topological space \( \langle X, \tau_d \rangle \), and mapping a nonexpansive/(uniformly)-continuous function \( f \) to itself, since it can be proven to be a continuous function between the corresponding topological spaces.

Example. Define \( d_S : B \times B \to \mathbb{R} \) such that \( d(t, f) \) maps to 1 and all other inputs map to 0. Then the evaluation of \( \tau(\langle B, d_S \rangle) \) is the Sierpiński space \( S \).