Kleisli Categories

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**Definition (Kleisli Category).** The Kleisli category of a monad \(\langle M : C \to C, \eta, \mu \rangle\), often denoted \(C_M\), is as follows:

**Objects** An object of \(C_M\) is an object \(C\) of \(C\).

**Morphisms** A morphism \(f : C \to C_M\) of \(C_M\) is a morphism \(f : C \to MC\) of \(C\).

**Identities** The \(C_M\)-identity of an object \(C\) is the \(C\)-morphism \(\eta_C : C \to MC\).

**Composition** The \(C_M\)-composition of \(f : C \to C_M\) and \(g : C_M \to C_M\) is the \(C\)-composition \(f ; M g ; \mu\).

These components satisfy the identity and associativity requirements of a category if and only if \(\eta\) and \(\mu\) satisfy the identity and associative laws of a monad.

There is also a (not necessarily faithful) “inclusion” functor \(I : C \to C_M\) that is the identity on objects and maps each \(C\)-morphism \(f : C \to C\) to the \(C_M\)-morphism corresponding to \(f ; \eta_C : C \to C\).

**Example.** The category \(\text{Set}_{\text{Option}}\) corresponds to sets with partial functions. The category \(\text{Set}_\text{M}\) corresponds to sets with enumerably-non-deterministic functions. The category \(\text{Set}_\text{f}\) corresponds to \(\text{Rel}\). The category \(\text{Set}_{s\to}\) is also known as the simple-slice category over \(S\). The category \(\text{Set}_{s\to s\times}\) corresponds to sets with \(S\)-stateful functions.

**Definition (Postmodule of a Monad).** Given a monad \(\langle m : C \to C, \eta, \mu \rangle\) of a 2-category \(C\), a postmodule, also known as a right module, is a 0-cell \(R\) along with a 1-cell \(r : C \to R\) and a 2-cell \(\rho : m \cdot r \Rightarrow r\) satisfying the following:

\[
\begin{array}{ccc}
C & \overset{m\cdot r}{\Rightarrow} & R \\
\downarrow^{\rho} & & \downarrow^{r} \\
C & \overset{r}{\Rightarrow} & R
\end{array}
\]

**Remark.** In terms of string diagrams, the above equalities are formulated as

\[
\begin{array}{ccc}
C & \overset{m\cdot r}{\Rightarrow} & R \\
\downarrow^{\rho} & & \downarrow^{r} \\
C & \overset{r}{\Rightarrow} & R
\end{array}
\]

\[
\begin{array}{ccc}
C & \overset{m\cdot r}{\Rightarrow} & R \\
\downarrow^{\rho} & & \downarrow^{r} \\
C & \overset{r}{\Rightarrow} & R
\end{array}
\]

**Example.** Every monad \(\langle m : C \to C, \eta, \mu \rangle\) is a postmodule of itself, with \(R = C\), \(r = m\), and \(\rho = \mu\).

**Example.** For any \(\text{Cat}\)-monad \(\langle M : C \to C, \eta, \mu \rangle\), the category \(C_M\) along with its inclusion functor \(I : C \to C_M\) forms a postmodule of \(\langle M, \eta, \mu \rangle\), with the components of the natural transformation \(\rho_C : MC \to MC\) given by the \(C\)-morphism \(id_MC : MC \to MC\). In fact, it is the \(\langle\text{co}\rangle\text{universal} \) postmodule of the monad \(\langle M, \eta, \mu \rangle\).

**Definition (Kleisli Object).** A Kleisli object \(C_m\) of a given monad \(\langle m : C \to C, \eta, \mu \rangle\) in a 2-category \(C\) is a \(\langle\text{co}\rangle\text{universal} \) postmodule of that monad.

**Remark.** Because every monad is its own postmodule, this implies there is a 1-cell \(f : C_m \to C\) (if \(C_m\) exists) such that \(i \cdot f = \text{id}\). One can show that these 1-cells always form an adjunction \(i \dashv f\) that gives rise to the monad \(m\).

For every adjunction \(\langle \eta, \varepsilon \rangle : \ell \dashv r : D \to C\) that gives rise to a monad \(\langle m, \eta, \mu \rangle\), the triple \((D, r, \varepsilon * r)\) is a premodule of \(\langle m, \eta, \mu \rangle\), and the triple \((D, \ell, \ell * \varepsilon)\) is a postmodule of \(\langle m, \eta, \mu \rangle\). Consequently, if \(C_m\) and \(C^m\) exist, then such an adjunction induces a sequence of 1-cells \(C_m \to D \to C^m\). Most notably, since \(C_m\) and \(C^m\) are each part of adjunctions that give rise to \(\langle m, \eta, \mu \rangle\), there is always a 1-cell from \(C_m\) to \(C^m\). In \(\text{Cat}\), this 1-cell describes the Kleisli category as a subcategory of the Eilenberg-Moore category comprised of the free monad algebras for \(\langle m, \eta, \mu \rangle\).