Kleisli Categories

Ross Tate

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Definition (Kleisli Category)**.** The Kleisli category of a monad $\langle M : \mathbf{C} \to \mathbf{C}, \eta, \mu \rangle$, often denoted $\mathbf{C}_M$, is as follows:

Objects An object of $\mathbf{C}_M$ is an object C of $\mathbf{C}$.

Morphisms A morphism $f : C \to_{\mathbf{C}_M} C'$ of $\mathbf{C}_M$ is a morphism $f : C \to_{\mathbf{C}} MC'$ of $\mathbf{C}$.

Identities The $\mathbf{C}_M$-identity of an object C is the $\mathbf{C}$-morphism $\eta_C : C \to_{\mathbf{C}} MC$.

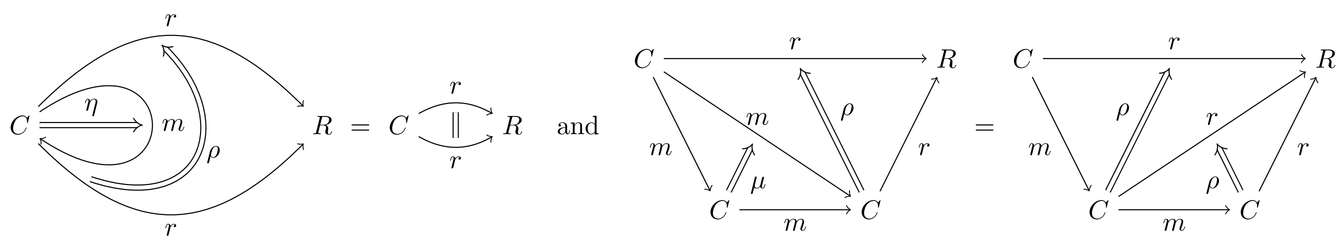
Composition The $\mathbf{C}_M$-composition of $f : C \to_{\mathbf{C}_M} C'$ and $g : C' \to_{\mathbf{C}_M} C''$ is the $\mathbf{C}$-composition $f \,;Mg\,; \mu_{C''}$.

These components satisfy the identity and associativity requirements of a category if and only if η and μ satisfy the identity and associative laws of a monad.

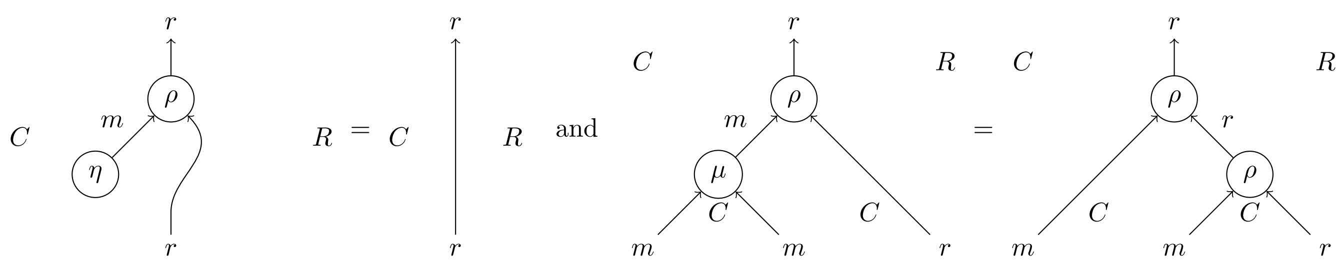
There is also a (not necessarily faithful) "inclusion" functor $I : \mathbf{C} \to \mathbf{C}_M$ that is the identity on objects and maps each $\mathbf{C}$-morphism $f : C \to_{\mathbf{C}} C'$ to the $\mathbf{C}_M$-morphism corresponding to $f \,; \eta_{C'} : C \to_{\mathbf{C}} MC'$.

Example. The category $\mathbf{Set}_{\text{Option}}$ corresponds to sets with partial functions. The category $\mathbf{Set}_{\mathbb{M}}$ corresponds to sets with enumerably-non-deterministic functions. The category $\mathbf{Set}_{\mathbb{P}}$ corresponds to $\mathbf{Rel}$. The category $\mathbf{Set}_{S\to\cdot}$ is also known as the simple-slice category over S. The category $\mathbf{Set}_{S\to S\times\cdot}$ corresponds to sets with S-stateful functions.

Definition (Postmodule of a Monad)**.** Given a monad $\langle m : C \to C, \eta, \mu \rangle$ of a 2-category $\mathbf{C}$, a postmodule, also known as a right module, is a 0-cell R along with a 1-cell $r : C \to R$ and a 2-cell $\rho : m\,; r \Rightarrow r$ satisfying the following:



Remark. In terms of string diagrams, the above equalities are formulated as



Example. Every monad $\langle m : C \to C, \eta, \mu \rangle$ is a postmodule of itself, with R as C, r as m, and ρ as μ.

Example. For any $\mathbf{Cat}$-monad $\langle M : \mathbf{C} \to \mathbf{C}, \eta, \mu \rangle$, the category $\mathbf{C}_M$ along with its inclusion functor $I : \mathbf{C} \to \mathbf{C}_M$ forms a postmosdule of $\langle M, \eta, \mu \rangle$, with the components of the natural transformation $\rho_C : MC \to_{\mathbf{C}_M} C$ given by the $\mathbf{C}$-morphism $id_{MC} : MC \to_{\mathbf{C}} MC$. In fact, it is the *(co)universal* postmodule of the monad $\langle M, \eta, \mu \rangle$.

Definition (Kleisli Object)**.** A Kleisli object C_m of a given monad $\langle m : C \to C, \eta, \mu \rangle$ in a 2-category $\mathbf{C}$ is a (co)universal postmodule of that monad.

Remark. Because every monad is its own postmodule, this implies there is a 1-cell $f : C_m \to C$ (if C_m exists) such that $i\,; f$ equals m. One can show that these 1-cells always form an adjunction $i \dashv f$ that gives rise to the monad m.

Remark. For every adjunction $\langle \eta, \varepsilon \rangle : \ell \dashv r : D \to C$ that gives rise to a monad $\langle m, \eta, \mu \rangle$, the triple $\langle D, r, \varepsilon * r \rangle$ is a premodule of $\langle m, \eta, \mu \rangle$, and the triple $\langle D, \ell, \ell * \varepsilon \rangle$ is a postmodule of $\langle m, \eta, \mu \rangle$. Consequently, if C_m and C^m exist, then such an adjunction induces a sequence of 1-cells $C_m \to D \to C^m$. Most notably, since C_m and C^m are each part of adjunctions that give rise to $\langle m, \eta, \mu \rangle$, there is always a 1-cell from C_m to C^m. In $\mathbf{Cat}$, this 1-cell describes the Kleisli category as a subcategory of the Eilenberg-Moore category comprised of the free monad algebras for $\langle m, \eta, \mu \rangle$.