Factorization Structures

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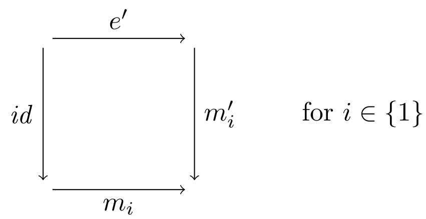
Definition (Factorization Structure)**.** We amend the definition of an $(\mathcal{E},\mathcal{M})$-factorization structure on sources with the requirement that every source $\{f_i\}_{i\in I}$ has an $(\mathcal{E},\mathcal{M})$-factoration $(e,\{m_i\}_{i\in I})$ *such that*, for all indices i and i' in I, f_i equals $f_{i'}$ implies m_i equals $m_{i'}$. This property is provable classically but not constructively.

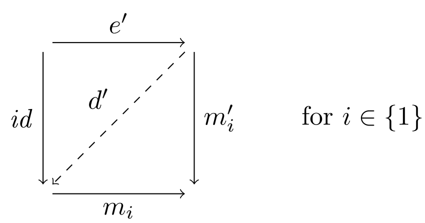
Lemma. *Given a category* $\mathbf{C}$ *with an* $(\mathcal{E},\mathcal{M})$*-factorization structure on sources, if a morphism* m *has the property that the unary source* $\langle m\rangle$ *is in* $\mathcal{M}$*, then the binary source* $\langle m,m\rangle$ *is also in* $\mathcal{M}$*.*

Proof. Define both m_1 and m_2 to be m. Let $(e',\langle m'_1,m'_2\rangle)$ be an $(\mathcal{E},\mathcal{M})$-factorization of the binary source $\langle m_1,m_2\rangle$ (with m'_1 equal to m'_2 since m_1 equals m_2). This means that the binary source $\langle m,m\rangle$, i.e. $\langle m_1,m_2\rangle$, is the composition of the binary $\mathcal{M}$-source $\langle m',m'\rangle$ with the morphism e. By the definition of factorization structure, the collection of sources $\mathcal{M}$ is closed under composition with isomorphisms. So if we can show that e' is an isomorphism, then the above facts imply that $\langle m,m\rangle$ is in $\mathcal{M}$.

To do so, note that the following indexed square commutes by construction of e' and m'_1 (and m'_2):

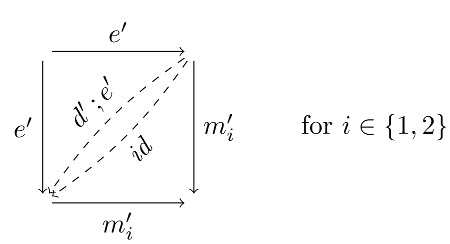


The top morphism e' belongs to $\mathcal{E}$ by construction, and the bottom unary source $\langle m_1\rangle$, i.e. $\langle m\rangle$, belongs to $\mathcal{M}$ by assumption. Consequently, by the definition of factorization structure, there exists a unique morphism d' making the following indexed diagram commute:



Thus we have a retraction d' of e', i.e. a morphism such that $e'\,;d'$ equals id. Furthermore, because m_1 equals m_2 *and* m'_1 equals m'_2, the morphism d' additionally has the property that $d\,;m_i$ equals m'_i for both $i=1$ *and* $i=2$.

Next we show that d' is in fact an inverse of e', i.e. that $d'\,;e'$ equals id. To do so, notice that the following indexed square has two diagonals that make everything commute, since $e'\,;d'\,;e' = id\,;e' = e'$ and $d'\,;e'\,;m'_i = d'\,;m_i = m'_i$:

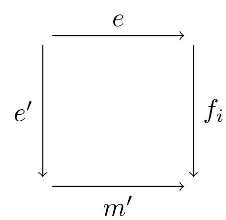


Since the top morphism belongs to $\mathcal{E}$ and the bottom source belongs to $\mathcal{M}$ (both by construction), diagonalizations of this square must be unique, which implies $d'\,;e'$ equals id. Thus e' is an isomorphism, with inverse d', making $\langle m,m\rangle$ an element of $\mathcal{M}$ by the reasoning above. □

Theorem. *Given a category* $\mathbf{C}$ *with an* $(\mathcal{E}, \mathcal{M})$*-factorization structure on sources, every morphism in* $\mathcal{E}$ *is epic.*

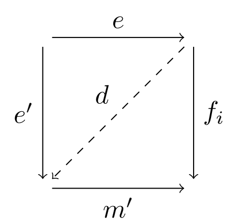
Proof. Suppose $e : A \to B$ is a morphism in $\mathcal{E}$, and suppose morphisms $f_1, f_2 : B \to C$ have the property that $e\,;f_1$ equals $e\,;f_2$. In order to prove e is epic, we must prove that f_1 equals f_2.

Let $(e', \langle m' \rangle)$ be an $(\mathcal{E}, \mathcal{M})$-factorization of $e\,;f_1$, or equivalently of $e\,;f_2$. Then by the above lemma, $\langle m', m' \rangle$ is also an element of $\mathcal{M}$. The following, then, is an indexed commuting square whose top morphism belongs to $\mathcal{E}$, by assumption, and whose bottom source belongs to $\mathcal{M}$, by the above lemma:



for $i \in \{1, 2\}$

Therefore there exists a morphism d such that the following indexed diagram commutes:



for $i \in \{1, 2\}$

Since this commutes for both $i = 1$ and $i = 2$, we get that $f_1 = d\,;m' = f_2$, thus implying that e is epic. ∎