Definition (Factorization Structure). We amend the definition of an \((\mathcal{E}, \mathcal{M})\)-factorization structure on sources with the requirement that every source \(\{f_i\}_{i \in I}\) has an \((\mathcal{E}, \mathcal{M})\)-factorization \((e, \{m_i\}_{i \in I})\) such that, for all indices \(i\) and \(i'\) in \(I\), \(f_i = f_{i'}\) implies \(m_i = m_{i'}\). This property is provable classically but not constructively.

Lemma. Given a category \(\mathcal{C}\) with an \((\mathcal{E}, \mathcal{M})\)-factorization structure on sources, if a morphism \(m\) has the property that the unary source \(\langle m \rangle\) is in \(\mathcal{M}\), then the binary source \(\langle m, m \rangle\) is also in \(\mathcal{M}\).

Proof. Define both \(m_1\) and \(m_2\) to be \(m\). Let \((e', \langle m'_1, m'_2 \rangle)\) be an \((\mathcal{E}, \mathcal{M})\)-factorization of the binary source \(\langle m_1, m_2 \rangle\) (with \(m'_1\) equal to \(m'_2\) since \(m_1 = m_2\)). This means that the binary source \(\langle m, m \rangle\), i.e. \(\langle m_1, m_2 \rangle\), is the composition of the binary \(\mathcal{M}\)-source \(\langle m', m' \rangle\) with the morphism \(e\). By the definition of factorization structure, the collection of sources \(\mathcal{M}\) is closed under composition with isomorphisms. So if we can show that \(e'\) is an isomorphism, then the above facts imply that \(\langle m, m \rangle\) is in \(\mathcal{M}\).

To do so, note that the following indexed square commutes by construction of \(e'\) and \(m'_1\) (and \(m'_2\)):

\[
\begin{array}{ccc}
id & & \langle m'_i \rangle \\
\downarrow & & \downarrow \circ \langle m_i \rangle \\
\langle m_i \rangle & & \langle m_i \rangle _{\{1\}}
\end{array}
\]

The top morphism \(e'\) belongs to \(\mathcal{E}\) by construction, and the bottom unary source \(\langle m_1 \rangle\), i.e. \(\langle m \rangle\), belongs to \(\mathcal{M}\) by assumption. Consequently, by the definition of factorization structure, there exists a unique morphism \(d'\) making the following indexed diagram commute:

\[
\begin{array}{ccc}
\langle m_i \rangle & & \langle m'_i \rangle _{\{1\}} \\
\downarrow & & \downarrow \circ \langle m_i \rangle \\
\langle m_i \rangle & & \langle m_i \rangle _{\{1\}}
\end{array}
\]

Thus we have a retraction \(d'\) of \(e'\), i.e. a morphism such that \(e' \circ d' = \text{id}\). Furthermore, because \(m_1 = m_2\) and \(m'_1 = m'_2\), the morphism \(d'\) additionally has the property that \(d' \circ m_i = m'_i\) for both \(i = 1\) and \(i = 2\).

Next we show that \(d'\) is in fact an inverse of \(e'\), i.e. that \(d' \circ e' = \text{id}\). To do so, notice that the following indexed square has two diagonals that make everything commute, since \(e' \circ d' = \text{id}\) and \(d' \circ e' = \text{id}\):

\[
\begin{array}{ccc}
e' & & m'_i _{\{1,2\}} \\
\downarrow \circ \langle m_i \rangle & & \downarrow \circ \langle m_i \rangle \\
\langle m_i \rangle & & \langle m_i \rangle _{\{1,2\}}
\end{array}
\]

Since the top morphism belongs to \(\mathcal{E}\) and the bottom source belongs to \(\mathcal{M}\) (both by construction), diagonalizations of this square must be unique, which implies \(d' \circ e' = \text{id}\). Thus \(e'\) is an isomorphism, with inverse \(d'\), making \(\langle m, m \rangle\) an element of \(\mathcal{M}\) by the reasoning above. \(\square\)
Theorem. Given a category $C$ with an $(\mathcal{E}, \mathcal{M})$-factorization structure on sources, every morphism in $\mathcal{E}$ is epic.

Proof. Suppose $e : A \to B$ is a morphism in $\mathcal{E}$, and suppose morphisms $f_1, f_2 : B \to C$ have the property that $e; f_1$ equals $e; f_2$. In order to prove $e$ is epic, we must prove that $f_1$ equals $f_2$.

Let $(e', (m'))$ be an $(\mathcal{E}, \mathcal{M})$-factorization of $e; f_1$, or equivalently of $e; f_2$. Then by the above lemma, $(m', m')$ is also an element of $\mathcal{M}$. The following, then, is an indexed commuting square whose top morphism belongs to $\mathcal{E}$, by assumption, and whose bottom source belongs to $\mathcal{M}$, by the above lemma:

\[
\begin{array}{ccc}
& & e \\
\downarrow e' & & \downarrow f_i \\
\downarrow m' & & \downarrow m' \\
& & \text{for } i \in \{1, 2\}
\end{array}
\]

Therefore there exists a morphism $d$ such that the following indexed diagram commutes:

\[
\begin{array}{ccc}
& & e \\
\downarrow e' & & \downarrow f_i \\
\downarrow e' & & \downarrow f_i \\
& & \text{for } i \in \{1, 2\}
\end{array}
\]

Since this commutes for both $i = 1$ and $i = 2$, we get that $f_1 = d; m' = f_2$, thus implying that $e$ is epic. \qed