Comma Categories

Ross Tate

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Definition. Given functors $\mathbf{A}_1 \xrightarrow{F_1} \mathbf{B} \xleftarrow{F_2} \mathbf{A}_2$, the comma category $F_1 \downarrow F_2$ is comprised of the following:

Objects A triple $\langle A_1 \in \mathrm{Ob}_{\mathbf{A}_1}, A_2 \in \mathrm{Ob}_{\mathbf{A}_1}, m \in \mathrm{Hom}_{\mathbf{B}}(F_1(A_1), F_2(A_2))\rangle$, often just written $F_1A_1 \xrightarrow{m} F_2A_2$.

Morphisms Given two objects $F_1A_1 \xrightarrow{m} F_2A_2$ and $F_1A_1' \xrightarrow{m'} F_2A_2'$, a morphism from m to m' is a pair $\langle f_1 \in \mathrm{Hom}_{\mathbf{A}_1}(A_1, A_1'), f_2 \in \mathrm{Hom}_{\mathbf{A}_2}(A_2, A_2')\rangle$ such that following square commutes:

$$\begin{array}{ccc} F_1A_1 & \xrightarrow{m} & F_2A_2 \\ {\scriptstyle F_1f_1}\downarrow & & \downarrow{\scriptstyle F_2f_2} \\ F_1A_1' & \xrightarrow{m'} & F_2A_2' \end{array}$$

Morphisms are often simply depicted by this square.

Identity The identity on object $m : F_1A_1 \to F_2A_2$ is the following:

$$\begin{array}{ccc} F_1A_1 & \xrightarrow{m} & F_2A_2 \\ {\scriptstyle F_1id_{A_1}}\downarrow & & \downarrow{\scriptstyle F_2id_{A_2}} \\ F_1A_1 & \xrightarrow{m} & F_2A_2 \end{array}$$

Composition The composition of morphims $\langle f_1, f_2\rangle$ and $\langle f_1', f_2'\rangle$ is the following:

$$\begin{array}{ccc} F_1A_1 & \xrightarrow{m} & F_2A_2 \\ {\scriptstyle F_1f_1}\downarrow & & \downarrow{\scriptstyle F_2f_2} \\ F_1A_1' & \xrightarrow{m'} & F_2A_2' \\ {\scriptstyle F_1f_1'}\downarrow & & \downarrow{\scriptstyle F_2f_2'} \\ F_1A_1'' & \xrightarrow{m''} & F_2A_2'' \end{array}$$

Definition. In the case where either F_1 or F_2 is actually the identity functor on $\mathbf{B}$, then one typically uses the notations $\mathbf{B} \downarrow F_2$ or $F_1 \downarrow \mathbf{B}$ rather than $\mathrm{Id}_{\mathbf{B}} \downarrow F_2$ or $F_1 \downarrow \mathrm{Id}_{\mathbf{B}}$. In general, as an abuse of notation, one often denotes the identity functor on a category with the category itself. Similarly, one often denotes the identity morphism on an object with the object itself.

Definition. $\mathbf{1}$ is the category with a single object $(\star)$ and a single morphism $(\star)$ on that object.

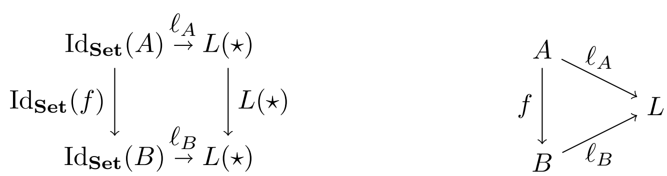
Example. Given functors $\mathbf{1} \xrightarrow{\mathbb{1}} \mathbf{Set} \xleftarrow{\mathrm{Id}_{\mathbf{Set}}} \mathbf{Set}$ (where the former is the constant functor picking out the singleton set $\mathbb{1}$), the comma category $\mathbb{1} \downarrow \mathbf{Set}$ is also known as **pSet**, the category of pointed sets. Unfolding definitions, an object in **pSet** is a set A and an element a of A. A morphism in **pSet** from $\langle A, a\rangle$ to $\langle B, b\rangle$ is a function $f : A \to B$ such that $f(a) = b$. In other words, the following diagrams commute:

$$\begin{array}{ccc} \mathbb{1}(\star) & \xrightarrow{a} & \mathrm{Id}_{\mathbf{Set}}(A) \\ {\scriptstyle \mathbb{1}(\star)}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}_{\mathbf{Set}}(f)} \\ \mathbb{1}(\star) & \xrightarrow{b} & \mathrm{Id}_{\mathbf{Set}}(B) \end{array} \qquad\qquad \begin{array}{ccc} & \overset{a}{\nearrow} & A \\ \mathbb{1} & & \downarrow{\scriptstyle f} \\ & \underset{b}{\searrow} & B \end{array}$$

Example. Given any category $\mathbf{A}$ and object A of $\mathbf{A}$, we can generalize the above construction with $A \downarrow \mathbf{A}$. This is also known as the category of objects *under* A, or as the *co*slice category ${}^{A}/_{\mathbf{A}}$.

Example. Given functors $\mathbf{Set} \xrightarrow{\mathrm{Id}_{\mathbf{Set}}} \mathbf{Set} \xleftarrow{\cdot^2} \mathbf{Set}$ (where the latter is the functor mapping a set A to the set of pairs A^2), the comma category $\mathbf{Set}\downarrow\cdot^2$ is (isomorphic to) **Graph**, the category of graphs. Unfolding definitions, an object in $\mathbf{Set}\downarrow\cdot^2$ is a set E, a set V, and a function from E to V^2 (i.e. $V \times V$), or equivalently a pair of functions $s, t : E \to V$. A morphism in $\mathbf{Set}\downarrow\cdot^2$ is a pair of functions $f_E : E \to E'$ and $f_V : V \to V'$ such that the following diagrams commutes:

Example. Given a set L and functors $\mathbf{Set} \xrightarrow{\mathrm{Id}_{\mathbf{Set}}} \mathbf{Set} \xleftarrow{L} \mathbf{1}$ (where the latter is the constant function picking out L), the comma category $\mathbf{Set}\downarrow L$ can be viewed as the category of sets with labeled elements and label-preserving functions. Unfolding definitions, an object in $\mathbf{Set}\downarrow L$ is a set A and a "labeling" function $\ell : A \to L$. A morphism in $\mathbf{Set}\downarrow L$ from $\langle A, \ell_A\rangle$ to $\langle B, \ell_B\rangle$ is a function $f : A \to B$ such that $\forall a \in A.\ \ell_B(f(a)) = \ell_A(a)$. In other words, the following diagrams commute:



Example. Given any category $\mathbf{A}$ and object A of $\mathbf{A}$, we can generalize the above construction with $\mathbf{A} \downarrow A$. This is also known as the category of objects *over* A, or as the slice category ${}^{\mathbf{A}}/_{A}$.

Definition. Given functors $\mathbf{A}_1 \xrightarrow{F_1} \mathbf{B} \xleftarrow{F_2} \mathbf{A}_2$, the functor $\pi_1 : F_1\downarrow F_2 \to \mathbf{A}_2$ maps the object $F_1A_1 \xrightarrow{m} F_2A_2$ to the object A_1 and the morphism $\langle f_1, f_2\rangle$ to the morphism f_1. Similarly, the functor $\pi_2 : F_1\downarrow F_2 \to \mathbf{A}_2$ maps the object $F_1A_1 \xrightarrow{m} F_2A_2$ to the object A_2 and the morphism $\langle f_1, f_2\rangle$ to the morphism f_2. Note that π_1 and π_2 are both abuses of notation and represent other constructs in other contexts. Also, sometimes they are instead denoted as $\pi_{\mathbf{A}_1}$ and $\pi_{\mathbf{A}_2}$.

Example. Given a set L and functors ${}^{\mathbf{Set}}/_{L} \xrightarrow{\pi_{\mathbf{Set}}} \mathbf{Set} \xleftarrow{\cdot^2} \mathbf{Set}$, the corresponding comma category is (isomorphic to) L-**Graph**, the category of graphs with L-labeled edges. Unfolding definitions, an object is a set E with a "labelling" function $\ell : E \to L$, a set V, and a function from E to V^2 (i.e. $V \times V$), or equivalently a pair of functions $s, t : E \to V$. A morphism is a morphism $f_\ell : \langle E, \ell\rangle \to \langle E', \ell'\rangle$ in ${}^{\mathbf{Set}}/_{L}$ and a function $f_V : V \to V'$ such that the following diagrams commutes (where $f_E = \pi_{\mathbf{Set}}(f_\ell)$):