Definition. Given functors $A_1 \xrightarrow{F_1} B \xleftarrow{F_2} A_2$, the comma category $F_1 \downarrow F_2$ is comprised of the following:

**Objects** A triple $(A_1 \in \text{Ob}_{A_1}, A_2 \in \text{Ob}_{A_1}, m \in \text{Hom}_B(F_1(A_1), F_2(A_2)))$, often just written $F_1 A_1 \xrightarrow{m} F_2 A_2$.

**Morphisms** Given two objects $F_1 A_1 \xrightarrow{m} F_2 A_2$ and $F_1 A_1' \xrightarrow{m'} F_2 A_2'$, a morphism from $m$ to $m'$ is a pair $\langle f_1 \in \text{Hom}_{A_1}(A_1, A_1'), f_2 \in \text{Hom}_{A_2}(A_2, A_2') \rangle$ such that the following square commutes:

\[
\begin{array}{ccc}
F_1 A_1 & \xrightarrow{m} & F_2 A_2 \\
F_1 f_1 & & \downarrow F_2 f_2 \\
F_1 A_1' & \xrightarrow{m'} & F_2 A_2'
\end{array}
\]

Morphisms are often simply depicted by this square.

**Identity** The identity on object $m : F_1 A_1 \to F_2 A_2$ is the following:

\[
\begin{array}{ccc}
F_1 A_1 & \xrightarrow{m} & F_2 A_2 \\
F_1 id_{A_1} & & \downarrow F_2 id_{A_2} \\
F_1 A_1 & \xrightarrow{m} & F_2 A_2
\end{array}
\]

**Composition** The composition of morphisms $\langle f_1, f_2 \rangle$ and $\langle f_1', f_2' \rangle$ is the following:

\[
\begin{array}{ccc}
F_1 A_1 & \xrightarrow{m} & F_2 A_2 \\
F_1 f_1 & \downarrow & \downarrow F_2 f_2 \\
F_1 A_1' & \xrightarrow{m'} & F_2 A_2' \\
F_1 f_1' & \downarrow & \downarrow F_2 f_2' \\
F_1 A_1'' & \xrightarrow{m''} & F_2 A_2''
\end{array}
\]

**Definition.** In the case where either $F_1$ or $F_2$ is actually the identity functor on $B$, then one typically uses the notations $B \downarrow F_2$ or $F_1 \downarrow B$ rather than $\text{Id}_B \downarrow F_2$ or $F_1 \downarrow \text{Id}_B$. In general, as an abuse of notation, one often denotes the identity functor on a category with the category itself. Similarly, one often denotes the identity morphism on an object with the object itself.

**Definition.** $\mathbb{1}$ is the category with a single object ($\star$) and a single morphism ($\star$) on that object.

**Example.** Given functors $\mathbb{1} \xrightarrow{\mathbb{1}} \text{Set} \xleftarrow{\text{Id}_{\text{Set}}} \text{Set}$ (where the former is the constant functor picking out the singleton set $\mathbb{1}$), the comma category $\mathbb{1} \downarrow \text{Set}$ is also known as $\text{pSet}$, the category of pointed sets. Unfolding definitions, an object in $\text{pSet}$ is a set $A$ and an element $a$ of $A$. A morphism in $\text{pSet}$ from $(A, a)$ to $(B, b)$ is a function $f : A \to B$ such that $f(a) = b$. In other words, the following diagrams commute:

\[
\begin{array}{ccc}
\mathbb{1}(\star) & \xrightarrow{a} & \text{Id}_{\text{Set}}(A) \\
\mathbb{1}(\star) & \xrightarrow{b} & \text{Id}_{\text{Set}}(B) \\
\mathbb{1}(\star) & \xrightarrow{a} & A \\
b & \xrightarrow{b} & B
\end{array}
\]
Example. Given any category $\mathbf{A}$ and object $A$ of $\mathbf{A}$, we can generalize the above construction with $A \downarrow \mathbf{A}$. This is also known as the category of objects under $A$, or as the coslice category $A / A$.

Example. Given functors $\mathbf{Set} \xrightarrow{\text{Id}_{\mathbf{set}}} \mathbf{Set} \xleftarrow{\ell} \mathbf{Set}$ (where the latter is the functor mapping a set $A$ to the set of pairs $A^2$), the comma category $\mathbf{Set} \downarrow \ell$ is (isomorphic to) $\mathbf{Graph}$, the category of graphs. Unfolding definitions, an object in $\mathbf{Set} \downarrow \ell$ is a set $E$, a set $V$, and a function from $E$ to $V^2$ (i.e. $E \times V$), or equivalently a pair of functions $s, t : E \to V$. A morphism in $\mathbf{Set} \downarrow \ell$ is a pair of functions $f_E : E \to E'$ and $f_V : V \to V'$ such that the following diagrams commute:

$$ \begin{array}{c}
\begin{array}{ccc}
\text{Id}_{\mathbf{Set}}(E) & \xrightarrow{(s, t)} & (V)^2 \\
\downarrow \text{Id}_{\mathbf{Set}}(f_E) & & \downarrow (f_V)^2 \\
\text{Id}_{\mathbf{Set}}(E') & \xrightarrow{(s', t')} & (V')^2
\end{array}
\end{array} \quad \quad \begin{array}{c}
\begin{array}{ccc}
V & \xrightarrow{s} & E & \xrightarrow{t} & V \\
\downarrow f_V & & \downarrow f_E & & \downarrow f_V \\
V' & \xleftarrow{s'} & E' & \xleftarrow{t'} & V'
\end{array}
\end{array} $$

Example. Given a set $L$ and functors $\mathbf{Set} \xrightarrow{\text{Id}_{L}} \mathbf{Set} \xleftarrow{\ell} \mathbf{Set}$ (where the latter is the constant function picking out $L$), the comma category $\mathbf{Set} \downarrow \ell L$ can be viewed as the category of sets with labeled elements and label-preserving functions. Unfolding definitions, an object in $\mathbf{Set} \downarrow \ell L$ is a set $A$ and a “labeling” function $\ell : A \to L$. A morphism in $\mathbf{Set} \downarrow \ell L$ from $\langle A, \ell_A \rangle$ to $\langle B, \ell_B \rangle$ is a function $f : A \to B$ such that $\forall a \in A. \ell_B(f(a)) = \ell_A(a)$. In other words, the following diagrams commute:

$$ \begin{array}{c}
\begin{array}{ccc}
\text{Id}_{\mathbf{Set}}(A) & \xrightarrow{\ell_A} & L(*) \\
\downarrow \text{Id}_{\mathbf{Set}}(f) & & \downarrow L(*) \\
\text{Id}_{\mathbf{Set}}(B) & \xrightarrow{\ell_B} & L(*)
\end{array}
\end{array} \quad \quad \begin{array}{c}
\begin{array}{ccc}
A & \xrightarrow{\ell_A} & L \\
\downarrow f & & \downarrow \ell_B \\
B & \xleftarrow{\ell_B}
\end{array}
\end{array} $$

Example. Given any category $\mathbf{A}$ and object $A$ of $\mathbf{A}$, we can generalize the above construction with $A \downarrow A$. This is also known as the category of objects over $A$, or as the slice category $\mathbf{A} / A$.

Definition. Given functors $\mathbf{A}_1 \xrightarrow{F_1} \mathbf{B} \xleftarrow{F_2} \mathbf{A}_2$, the functor $\pi_1 : F_1 \downarrow F_2 \rightarrow \mathbf{A}_2$ maps the object $F_1 A_1 \xrightarrow{m} F_2 A_2$ to the object $A_1$ and the morphism $\langle f_1, f_2 \rangle$ to the morphism $f_1$. Similarly, the functor $\pi_2 : F_1 \downarrow F_2 \rightarrow \mathbf{A}_2$ maps the object $F_1 A_1 \xrightarrow{m} F_2 A_2$ to the object $A_2$ and the morphism $\langle f_1, f_2 \rangle$ to the morphism $f_2$. Note that $\pi_1$ and $\pi_2$ are both abuses of notation and represent other constructs in other contexts. Also, sometimes they are instead denoted as $\pi_{A_1}$ and $\pi_{A_2}$.

Example. Given a set $L$ and functors $\mathbf{Set}/L \xrightarrow{\pi_{\mathbf{Set}} \sim \mathbf{Set}} \mathbf{Set} \xleftarrow{\ell} \mathbf{Set}$, the corresponding comma category is (isomorphic to) $L$-Graph, the category of graphs with $L$-labeled edges. Unfolding definitions, an object is a set $E$ with a “labelling” function $\ell : E \to L$, a set $V$, and a function from $E$ to $V^2$ (i.e. $E \times V$), or equivalently a pair of functions $s, t : E \to V$. A morphism is a morphism $f_E : (E, \ell) \to (E', \ell')$ in $\mathbf{Set}/L$ and a function $f_V : V \to V'$ such that the following diagrams commute (where $f_E = \pi_{\mathbf{Set}}(f_E)$):

$$ \begin{array}{c}
\begin{array}{ccc}
\pi_{\mathbf{Set}}((E, \ell)) & \xrightarrow{(s, t)} & (V)^2 \\
\downarrow \pi_{\mathbf{Set}}(f_E) & & \downarrow (f_V)^2 \\
\pi_{\mathbf{Set}}((E', \ell')) & \xrightarrow{(s', t')} & (V')^2
\end{array}
\end{array} \quad \quad \begin{array}{c}
\begin{array}{ccc}
V & \xleftarrow{s} & E & \xrightarrow{t} & V \\
\downarrow f_V & & \downarrow f_E & & \downarrow f_V \\
V' & \xleftarrow{s'} & E' & \xleftarrow{t'} & V'
\end{array}
\end{array} $$
