Definition. Given a functor \( T : X \to X \), the concrete category over \( X \) of \( T \)-coalgebras \( \text{Coalg}(T) \) is comprised of the following:

**Objects** An object \( \langle X, c \rangle \) is a pair of an (underlying) object \( X \) of \( X \) and an \( X \)-morphism \( c : X \to T(X) \).

**Morphisms** A morphism from \( \langle X, c \rangle \) to \( \langle X', c' \rangle \) is an (underlying) \( X \)-morphism \( f : X \to X' \) such that the following commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{c} & T(X) \\
\downarrow{f} & & \downarrow{T(f)} \\
X' & \xrightarrow{c'} & T(X')
\end{array}
\]

Being a concrete category over \( X \), identity and composition are inherited from \( X \). Identities can easily be shown to make the square commute, and composition can easily be shown to preserve commutation of squares, so this is a well-defined category (concrete over \( X \)).

Example. Given a set \( \Sigma \), the function on sets \( \lambda X. X^{\Sigma} \times B \) extends to an endofunctor on \( \text{Set} \) by mapping a function \( f \) to the function \( \lambda(x, b). (\lambda \sigma. f(x(\sigma))), b \). A coalgebra of this functor is a set \( S \) and a function of the form \( S \to S^{\Sigma} \times B \). Note that such a function corresponds to a pair of functions \( S \to S^{\Sigma} \) and \( S \to B \). The former further corresponds to a function \( S \times \Sigma \to S \), and the latter corresponds to a (decidable) subset of \( S \). This a coalgebra of this functor is a set (of states) \( S \), a (transition) function \( \delta : S \times \Sigma \to S \), and a (decidable) subset of (accepting) states. In other words, an object of \( \text{Coalg}(\Sigma^{\times} \times B) \) is essentially a \( \Sigma \)-acceptor without an initial state, and a morphism of \( \text{Coalg}(\Sigma^{\times} \times B) \) is essentially a morphism of \( \Sigma \)-acceptors that preserves and reflects transitions and accepting states.

Example. Let \( \text{Fin} \) be the category of finite sets. An object of \( \text{Coalg}(\mathcal{P}(\cdot)^{\Sigma} \times B) \) (concrete over \( \text{Fin} \)) is a non-deterministic finite automaton without an initial state. A morphism of \( \text{Coalg}(\mathcal{P}(\cdot)^{\Sigma} \times B) \) (concrete over \( \text{Fin} \)) is a morphism of non-deterministic finite automata that preserves and reflects transitions and accepting states.

Example. Let \( B \) be an abstract symbol denoting “blank”, and let \( L \) and \( R \) be abstract symbols denoting “left” and “right”. An object of \( \text{Coalg}(\text{Option}(\cdot^{\times} \Sigma \times \{L, R\})^{\Sigma^{+}(B)}) \) (concrete over \( \text{Fin} \)) is a Turing machine without an initial state, and a morphism of \( \text{Coalg}(\text{Option}(\cdot^{\times} \Sigma \times \{L, R\})^{\Sigma^{+}(B)}) \) is a morphism of Turing machines that preserves and reflects transitions, outputs, movements, and haltings.