Assignment 8

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Exercise 1. **Set** has a cool property. Suppose you have a commuting square

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, hook, "m"'] & D \end{tikzcd}$$

where e is a surjection and m is an injection. Then there exists a (unique) diagonal $d : B \to C$ such that the following commutes:

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dl, dashed, "d"'] \\ C \arrow[r, hook, "m"'] & D \end{tikzcd}$$

This function is given by $d(b) = f(a)$ for any $a \in A$ such that $e(a) = b$. The reason this function is well-defined is two-fold. First, because e is surjective, for every input $b \in B$ there necessarily exists some $a \in B$ such that $e(a) = b$, thereby making d total. Second, for any two a and a' in A such that $e(a) = e(a') = b$, $f(a)$ necessarily equals $f(a')$, making d determined. The reason is that m is injective, so $f(a) = f(a')$ holds if $m(f(a)) = m(f(a'))$ holds, and the latter is equivalent to $g(e(a)) = g(e(a'))$ because the square commutes, which is then equivalent to $g(b) = g(b)$ because of the assumed equalities, which clearly holds by reflexivity.

From a categorical perspective, this proof is actually just applying the fact that every surjection is a regular epimorphism, every injection is a monomorphism, and every category has (unique) (RegEpi,Mono)-diagonalizations. This last property of a given category $\mathbf{C}$ means that, given any commuting square as above such that e is a regular epimorphism and m is a monomorphism in $\mathbf{C}$, then there exists a (unique) commuting diagonal d as above. Prove that every category has (RegEpi,Mono)-diagonalizations. (Interesting side note: this in fact generalizations to when g is a source and m is a monosource.)

Exercise 2. Suppose the I-indexed source $\{P \xrightarrow{p_i} A\}_{i \in I}$ is a product. Note that the codomain of all these morphisms is the same: A. Prove that p_i is a retract for every $i \in I$.