Exercise 1. Set has a cool property. Suppose you have a commuting square

\[
\begin{array}{c}
A \\ e \\ f \\ C \\
\downarrow \\
 B \\ g \\ D
\end{array}
\]

where \(e\) is a surjection and \(m\) is an injection. Then there exists a (unique) diagonal \(d : B \rightarrow C\) such that the following commutes:

\[
\begin{array}{c}
A \\ e \\ f \\ C \\
\downarrow \\
 B \\ d \\ g \\ D
\end{array}
\]

This function is given by \(d(b) = f(a)\) for any \(a \in A\) such that \(e(a) = b\). The reason this function is well-defined is two-fold. First, because \(e\) is surjective, for every input \(b \in B\) there necessarily exists some \(a \in B\) such that \(e(a) = b\), thereby making \(d\) total. Second, for any two \(a\) and \(a'\) in \(A\) such that \(e(a) = e(a') = b\), \(f(a)\) necessarily equals \(f(a')\), making \(d\) determined. The reason is that \(m\) is injective, so \(f(a) = f(a')\) holds if \(m(f(a)) = m(f(a'))\) holds, and the latter is equivalent to \(g(e(a)) = g(e(a'))\) because the square commutes, which is then equivalent to \(g(b) = g(b)\) because of the assumed equalities, which clearly holds by reflexivity.

From a categorical perspective, this proof is actually just applying the fact that every surjection is a regular epimorphism, every injection is a monomorphism, and every category has (unique) (RegEpi,Mono)-diagonalizations. This last property of a given category \(C\) means that, given any commuting square as above such that \(e\) is a regular epimorphism and \(m\) is a monomorphism in \(C\), then there exists a (unique) commuting diagonal \(d\) as above. Prove that every category has (RegEpi,Mono)-diagonalizations. (Interesting side note: this in fact generalizations to when \(g\) is a source and \(m\) is a monosource.)

Exercise 2. Suppose the \(I\)-indexed source \(\{P \overrightarrow{p_i} A\}_{i \in I}\) is a product. Note that the codomain of all these morphisms is the same: \(A\). Prove that \(p_i\) is a retract for every \(i \in I\).