Assignment 11

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Definition (Multilinear Map)**.** A multilinear map from a list of commutative monoids $\langle A_1, 0_1, +_1\rangle, \ldots, \langle A_n, 0_n, +_n$ to a commutative monoid $\langle B, 0, +\rangle$ is a n-ary function $f : A_1 \times \cdots \times A_n \to B$ satisfying the following properties for any index $i \in \{1, \ldots, n\}$ and elements $a_1 \in A_1, \ldots, a_{i-1} \in A_{i-1}, a_{i+1} \in A_{i+1}, \ldots, a_n \in A_n$:

$$f(a_1, \ldots, 0_i, \ldots, a_n) = 0 \quad \text{and} \quad \forall a_i, a'_i \in A_i.\ f(a_1, \ldots, a_i +_i a'_i, \ldots, a_n) = f(a_1, \ldots, a_i, \ldots, a_n) + f(a_i, \ldots, a'_i, \ldots, a_n)$$

That is, a multilinear map is an n-ary function such that, for every index i, fixing all the inputs *besides* the input for i results in a monoid homomorphism from $\langle A_i, 0_i, +_i\rangle$ to $\langle B, 0, +\rangle$. In particular, a unary multilinear map is simply a monoid homomorphism, and a nullary multilinear is simply a nullary function (i.e. an element of the codomain).

Exercise 1. **CommMon** is the multicategory whose objects are commutative monoids and whose morphisms are multilinear maps. Its identities and compositions are inherited from **Set**. Show that all internal monoids of **CommMon** specify a semiring (defined in the previous homework). All semirings specify an internal monoid of **CommMon** as well, and this correspondence is bijective, but do not prove that fact.

Definition (Strong Endofunctor)**.** A strong endofunctor T of a multicategory $\mathbf{C}$ is a function on the objects of $\mathbf{C}$ along with, for every multimorphism $f : [A_1, \ldots, A_n] \to B$ of $\mathbf{C}$ and index $i \in \{1, \ldots, n\}$, a multimorphism $T(f, i) : [A_1, \ldots, T(A_i), \ldots, A_n] \to TB$ of $\mathbf{C}$, such that the following properties hold:

$$\forall A. \quad T(id_A, 1) = id_{TA}$$

$$\forall \begin{array}{ll} f_1 : [A^1_1, \ldots, A^1_{m_1}] \to B_1, & g : [B_1, \ldots, B_n] \to C, \\ \vdots & i \in \{1, \ldots, n\}, \\ f_n : [A^n_1, \ldots, A^n_{m_n}] \to B_n, & j \in \{1, \ldots, m_i\}. \end{array} \quad T\left(\mathop{\triangle}\limits_{f_1,\ldots,f_n} g, j + \sum_{k \in \{1,\ldots,i-1\}} m_k\right) = \mathop{\triangle}\limits_{f_1,\ldots,T(f_i,j),\ldots,f_n} T(g, i)$$

Example. Given a set S, the function on sets $S \times \cdot$ extends to a strong endofunctor on **Set**. Note that there is no way to extend $S \times \cdot$ to an endomultifunctor on **Set** (without some additional structure on S, such as a monoid on S).

Definition (Natural Transformation)**.** A natural transformation α from a strong endofunctors F to a strong endofunctor G on $\mathbf{C}$ is a collection of multimorphisms $\{\alpha_C : [FC] \to GC\}_{C \in \mathbf{C}}$ such that the following property holds:

$$\forall f : [A_1, \ldots, A_n] \to B, i \in \{1, \ldots, n\}. \quad \mathop{\triangle}\limits_{id_{A_1},\ldots,\alpha_{A_i},\ldots,id_{A_n}} G(f, i) = \mathop{\triangle}\limits_{F(f,i)} \alpha_B$$

Example. Given a function f from a set S to a set T, there is a corresponding natural transformation from the strong endofunctor $S \times \cdot$ to the strong endofunctor $T \times \cdot$ on **Set**.

Definition (Strong Monad)**.** A strong monad on a multicategory $\mathbf{C}$ is an internal monoid of $\mathbf{Strong}(\mathbf{C})$, which is comprised of the following:

Objects An object is a strong endofunctor of $\mathbf{C}$

Multimorphisms A multimorphism from $[F_1, \ldots, F_n]$ to G is a natural transformation from $F_1\,;\ldots;F_n$ to G, where $F_1\,;\ldots;F_n$ is the following strong functor:

$$(F_1\,;\ldots;F_n)(C) = F_n(\ldots F_1(C))) \qquad (F_1\,;\ldots;F_n)(f, i) = F_n(\ldots F_1(f, i), i)$$

Identities The identity id_F is the collection $\{id_{FC}\}_{C \in \mathbf{C}}$

Compositions The composition $\triangle_{\alpha^1 : \vec{F}_1 \to G_1, \ldots, \alpha^n : \vec{F}_n \to G_n} \beta$ is the collection $\{\alpha^n_C\,; G_n(\ldots \alpha^1_C, 1)\,; \beta_C\}_{C \in \mathbf{C}}$

Example. Lists $\mathbb{L}$ forms a strong monad on **Set**. Given a function $f : [A_1, \ldots, A_n] \to B$ and index $i \in \{1, \ldots, n\}$, one defines $\mathbb{L}(f, i)(a_1, \ldots, [a^1_i, \ldots, a^m_i], \ldots, a_n)$ as $[f(a_1, \ldots, a^1_i, \ldots, a_n), \ldots, f(a_1, \ldots, a^m_i, \ldots, a_n)]$. Its unit η is the singleton transformation, and its join μ is the flatten transformation. A similar construction forms a strong monad on **Set** from multisets/bags $\mathbb{M}$.

Definition (Eilenberg-Moore Multicategory of a Strong Monad)**.** Given a strong monad $\langle M, \eta, \mu \rangle$ on a multicategory $\mathbf{C}$, its Eilenberg-Moore multicategory is comprised of the following (with identities and compositions inherited from $\mathbf{C}$):

Objects An object is an object A of $\mathbf{C}$ along with a multimorphism $a : [MA] \to A$ satisfying the following:

$$id_A = \mathop{\triangle}\limits_{\eta_A} a \qquad \mathop{\triangle}\limits_{M(a,1)} a = \mathop{\triangle}\limits_{\mu_A} a$$

Multimorphisms A multimorphism from $[\langle A_1, a_1 \rangle, \ldots, \langle A_n, a_n \rangle]$ to $\langle B, b \rangle$ is a multimorphism f of $\mathbf{C}$ from the list of objects $[A_1, \ldots, A_n]$ to B satisfying the following *multilinearity* property:

$$\forall i \in \{1, \ldots, n\}. \quad \mathop{\triangle}\limits_{T(f,i)} b = \mathop{\triangle}\limits_{id_{A_1},\ldots,a,\ldots,id_{A_n}} f$$

Example. **Mon** is the Eilenberg-Moore multicategory of the strong monad $\mathbb{L}$ on **Set**. **CommMon** is the Eilenberg-Moore multicategory of the strong monad $\mathbb{M}$ on **Set**.

Definition (Unit)**.** An object I with a multimorphism $\mathsf{unit} : [\,] \to I$ is called a unit object with a unit multimorphism if they form a tensor of the empty list. A multicategory has a unit if it has such an object and multimorphism.

Exercise 2. Prove that $\langle \mathbb{N}, 0, + \rangle$ is a unit object of **CommMon**. However, rather than showing the existence and uniqueness of $\mathsf{split}_{\vec{A};\varnothing;\vec{B}} f$ for arbitray lists of commutative monoids $\vec{A}$ and $\vec{B}$, for the sake of readability show this only for the case where $\vec{A}$ is a singleton list and $\vec{B}$ is empty.

Remark. Note that $\langle \mathbb{N}, 0, + \rangle$ is *not* a unit object of **Mon**. In fact, **Mon** has no unit. The construction in the proof of the above only generalizes to *commutative* strong monads on multicategories with a unit, meaning strong monads with the following additional property:

$$\forall f : [A_1, \ldots, A_n] \to B.\ \forall i, j \in \{1, \ldots, n\}. \qquad \mathop{\bigtriangleup}\limits_{M(M(f,i),j)} \mu_B = \mathop{\bigtriangleup}\limits_{M(M(f,j),i)} \mu_B$$

Note that one can extend a commutative strong monad into a multifunctor monad through the following construction:

$$(f : [A_1, \ldots, A_n] \to B) \mapsto \left(\mathop{\bigtriangleup}\limits_{M(\ldots M(f,1)\ldots,n)} \mu_n \right) : [MA_1, \ldots, MA_n] \to MB$$

Exercise 3. Prove that, for any given commutative monoids $\langle A, 1, * \rangle$ and $\langle B, 1, * \rangle$, the set of monoid homomorphisms from $\langle A, 1, * \rangle$ to $\langle B, 1, * \rangle$ is the underlying set of the left-exponential object $\langle A, 1, * \rangle \multimap \langle B, 1, * \rangle$ in **CommMon**. However, for the existence and uniqueness proof of λf, for the sake of readability only show this for the case where f is binary.

Remark. Note that the set of monoid homomorphisms fails to be the underlying set of a left-exponential object in **Mon**. In fact, **Mon** only has exponentials when $\langle B, 1, * \rangle$ is commutative. More generally, the construction in the proof of the above once again only generalizes to *commutative* strong monads on left-closed multicategories.

Remark. One might notice that there is no exercise for binary tensors, only nullary tensors. This is because binary tensors require significantly more structure on both the underlying multicategory and the strong monad. In **CommMon**, the underlying set of the tensor of $\langle A, 0, + \rangle$ and $\langle B, 0, + \rangle$ is the set ${}^{\mathbb{M}(A \times B)}/_{\approx}$, where $\approx$ is the least equivalence relation satisfying:

$$[\langle 0, b \rangle] \approx [\,] \qquad [\langle a + a', b \rangle] \approx [\langle a, b \rangle, \langle a', b \rangle] \qquad [\langle a, 0 \rangle] \approx [\,] \qquad [\langle a, b + b' \rangle] \approx [\langle a, b \rangle, \langle a, b' \rangle]$$

$$\ell_1 \approx \ell_1' \wedge \ell_2 \approx \ell_2' \implies \ell_1 \mathbin{+\!\!+} \ell_2 \approx \ell_1' \mathbin{+\!\!+} \ell_2' \qquad\qquad (\ell \mathbin{+\!\!+} \ell' \approx \ell' \mathbin{+\!\!+} \ell)$$

More generally, the multicategory must have strong coequalizers. Furthermore, the monad must provide strong congruence quotients, meaning for any epimorphism $e : [MA] \to B$ there exists a monad algebra $q : [MQ] \to Q$ and a morphism $e_q : B \to Q$ such that $\triangle_{\mu_A} \triangle_e e_q = \triangle_{M(\triangle_e e_q, 1)} q$, and furthermore for any monad algebra $f : MF \to F$ and multimorphism $m : [C_1, \ldots, C_n] \mathbin{+\!\!+} [B] \mathbin{+\!\!+} \vec{D} \to F$ such that $\triangle_{M(m, n+1)} f = \triangle_{\triangle_{\mu_A} e} m$ there exists a unique multimorphism $m' : [C_1, \ldots, C_n] \mathbin{+\!\!+} [Q] \mathbin{+\!\!+} \vec{D} \to F$ such that $\triangle_{id_{C_1}, \ldots, id_{C_n}, e_q, \overrightarrow{id_D}} f' = f$.