Kan Extensions

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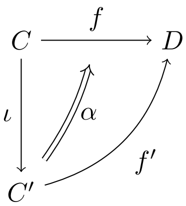
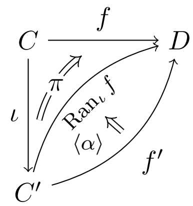
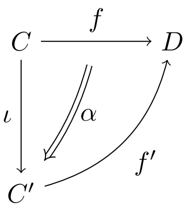
Definition (Kan Extension)**.** Given a 2-category, a *right* Kan extension of a 1-cell $f : C \to D$ along a 1-cell $\iota : C \to C'$ is a 1-cell, typically denoted $\mathrm{Ran}_\iota\, f : C' \to D$, along with a 2-cell $\pi : \iota\, ; \mathrm{Ran}_\iota\, f \Rightarrow f : C \to D$ that is *universal*:

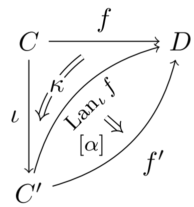
Given any 

, there exists a unique 2-cell $\langle\alpha\rangle : f' \Rightarrow \mathrm{Ran}_\iota$ such that 

equals α.

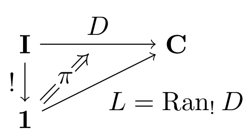
On the flipside, a *left* Kan extension of $f : C \to D$ along $\iota : C \to C'$ is a 1-cell, typically denoted $\mathrm{Lan}_\iota\, f : C' \to D$, along with a 2-cell $\kappa : f \Rightarrow \iota\, ; \mathrm{Lan}_\iota\, f : C \to D$ that is *(co)universal*:

Given any 

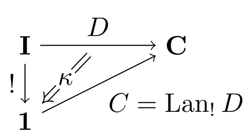
, there exists a unique 2-cell $[\alpha] : \mathrm{Lan}_\iota \Rightarrow f'$ such that 

equals α.

Example. Let $\mathbf{I}$ be a category representing a scheme, and let $D : \mathbf{I} \to \mathbf{C}$ be a diagram with scheme $\mathbf{I}$ in category $\mathbf{D}$. Then a limit $L \in \mathbf{C}$ of this diagram with natural source $\{L \xrightarrow{\pi_I} DI\}_{I \in \mathbf{I}}$ is a right Kan extension of D along the unique functor from $\mathbf{C}$ to the singleton category $\mathbf{1}$:



Similarly, a colimit $C \in \mathbf{C}$ of this diagram with natural sink $\{DI \xrightarrow{\kappa_I} C\}_{I \in \mathbf{I}}$ is a left Kan extension of D along the unique functor from $\mathbf{C}$ to the singleton category $\mathbf{1}$:



Definition (Absolute Kan Extension)**.** A right/left Kan extension is absolute if, for all 1-cells $g : D \to E$, the composition $(\mathrm{Ran}_\iota\, f)\, ; g/(\mathrm{Lan}_\iota\, f)\, ; g$ is a right/left Kan extension of $f\, ; g$ along ι with corresponding 2-cell $\pi * g/\kappa * g$.

Example. A 1-cell $\ell : C \to D$ is a left adjoint if and only if $\mathrm{Lan}_\ell\, id_C$ exists and is absolute, in which case $\mathrm{Lan}_\ell\, id_C$ is the corresponding right adjoint r and the universal 2-cell from id_C to $\ell\, ; \mathrm{Lan}_\ell\, id_C$ is the unit of the adjunction $\eta : C \Rightarrow \ell\, ; r$. The counit of the adjunction $\varepsilon : r\, ; \ell \Rightarrow D$ is given by the fact that $\eta * \ell : id_C\, ; \ell \Rightarrow \ell\, ;(r\, ; \ell)$ is also (co)universal due to absoluteness and id_ℓ is a 2-cell from $id_C\, ; \ell$ to $\ell\, ; id_D$, which implies a unique corresponding 2-cell from $r\, ; \ell$ to id_D.

On the flipside, a 1-cell $r : C \to D$ is a right adjoint if and only if $\mathrm{Ran}_r\, id_C$ exists and is absolute. Note that these formulate left/right adjoints as properties of a 1-cell and imply that the remaining components of an adjunction are determined uniquely up to isomorphism.