Coalgebras

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Definition. Given an functor $T : \mathbf{X} \to \mathbf{X}$, the concrete category over $\mathbf{X}$ of T-coalgebras $\mathbf{Coalg}(T)$ is comprised of the following:

Objects An object $\langle X, c\rangle$ is a pair of an (underlying) object X of $\mathbf{X}$ and an $\mathbf{X}$-morphism $c : X \to T(X)$.

Morphisms A morphism from $\langle X, c\rangle$ to $\langle X', c'\rangle$ is an (underlying) $\mathbf{X}$-morphism $f : X \to X'$ such that the following commutes:

$$\begin{array}{ccc} X & \xrightarrow{c} & T(X) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle T(f)} \\ X' & \xrightarrow[c']{} & T(X') \end{array}$$

Being a concrete category over $\mathbf{X}$, identity and composition are inherited from $\mathbf{X}$. Identities can easily be shown to make the square commute, and composition can easily be shown to preserve commutation of squares, so this is a well-defined category (concrete over $\mathbf{X}$).

Example. Given a set Σ, the function on sets $\lambda X.\, X^\Sigma \times \mathbb{B}$ extends to an endofunctor on $\mathbf{Set}$ by mapping a function f to the function $\lambda\langle x, b\rangle.\, \langle(\lambda\sigma.\, f(x(\sigma))), b\rangle$. A coalgebra of this functor is a set S and a function of the form $S \to S^\Sigma \times \mathbb{B}$. Note that such a function corresponds to a pair of functions $S \to S^\Sigma$ and $S \to \mathbb{B}$. The former further corresponds to a function $S \times \Sigma \to S$, and the latter corresponds to a (decidable) subset of S. This a coalgebra of this functor is a set (of states) S, a (transition) function $\delta : S \times \Sigma \to S$, and a (decidable) subset of (accepting) states. In other words, an object of $\mathbf{Coalg}(\cdot^\Sigma \times \mathbb{B})$ is essentially a Σ-acceptor without an initial state, and a morphism of $\mathbf{Coalg}(\cdot^\Sigma \times \mathbb{B})$ is essentially a morphism of Σ-acceptors that preserves *and reflects* transitions and accepting states.

Example. Let $\mathbf{Fin}$ be the category of finite sets. An object of $\mathbf{Coalg}(\mathbb{P}(\cdot)^\Sigma \times \mathbb{B})$ (concrete over $\mathbf{Fin}$) is a non-deterministic finite automaton without an initial state. A morphism of $\mathbf{Coalg}(\mathbb{P}(\cdot)^\Sigma \times \mathbb{B})$ (concrete over $\mathbf{Fin}$) is a morphism of non-deterministic finite automata that preserves *and reflects* transitions and accepting states.

Example. Let B be an abstract symbol denoting "blank", and let L and R be abstract symbols denoting "left" and "right". An object of $\mathbf{Coalg}(\mathrm{Option}(\cdot \times \Sigma \times \{L, R\})^{\Sigma+\{B\}})$ (concrete over $\mathbf{Fin}$) is a Turing machine without an initial state, and a morphism of $\mathbf{Coalg}(\mathrm{Option}(\cdot \times \Sigma \times \{L, R\})^{\Sigma+\{B\}})$ is a morphism of Turing machines that preserves *and reflects* transitions, outputs, movements, and haltings.