Exercise 1. Prove that for any 2-category $C$ and any adjunction $f \dashv g$ in $C$, one can build a monad in $C$ whose underlying morphism is $f : g$.

Proof. Let $f$ be from $C$ to $C$, and let $\eta$ and $\varepsilon$ be the unit and counit of the adjunction. Then $\langle C, f : g, \mu, a, \eta, i \rangle$ is an adjunction, where $\mu$, $d$, and $i$ are defined as follows:

- $\mu : f : g : f : g \Rightarrow f : g = f g f g \varepsilon$
- $a$ is given by the fact that both compositions result in the following string diagram:

![](image1.png)

- $i$ is given by $f g \eta \varepsilon$ and $f g \varepsilon \eta$, and equal due to adjunction properties.

Exercise 2. Prove that, in the 2-category $\text{CAT}$, for every monad $M$ with underlying functor $M$ on a category $C$ there is some adjunction $F \dashv U$ such that $M$ equals $F ; U$. Hint: use the underlying functor $U : \text{Alg}(M) \to C$ as the right adjoint.

Proof. Let $M$ be $\langle C, M, \mu, a, \eta, i \rangle$. Let $U : \text{Alg}(M) \to C$ be the underlying functor of $\text{Alg}(M)$. Let $F : C \to \text{Alg}(M)$ be the functor mapping each object $C$ to the algebra $\langle M(C), \mu_C, a, i \rangle$ and each morphism $f : C_1 \to C_2$ to the algebra morphism $\langle M(f), \delta_f \rangle$ where $\delta_f : M(M(f)) \mu_{C_2} = \mu_{C_1} ; M(f)$ comes from naturality of $\mu$. The fact that $F$ is functorial comes from functoriality of $M$. $F ; U$ then equals $M$, so we can define the unit of the adjunction $\eta : C \Rightarrow F ; U$ as the unit of the monad $\eta : C \Rightarrow M$. For the counit $\varepsilon$, we map each algebra $\langle C, a, \delta_a, i \rangle$ to the morphism of algebras $\langle a, \delta_a : U(F(\langle C, a, \delta_a, i \rangle)) = \langle M(C), \mu_C, a, i \rangle \to \langle C, a, \delta_a, i \rangle \rangle$. $\varepsilon$ is natural because all algebra morphisms are distributive. Lastly, $(\eta \cdot F ; (F \cdot \varepsilon))_C$ is defined as $M(\eta_C) ; \mu_C$ which equals the identity since $\eta$ is an identity of $\mu$, and $(U \cdot \eta ; (\varepsilon \cdot U))_{C(a, \eta, i)}$ is defined as $\eta_C ; a$ which equals the identity by $i_a$. Thus, $\langle C, F, U, \eta, \varepsilon, i \rangle$ forms an adjunction with $F ; U$ equal to $M$. □