Monads

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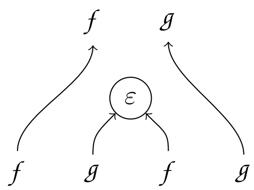
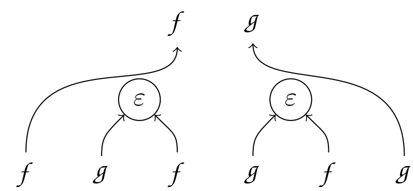
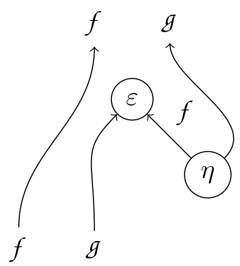
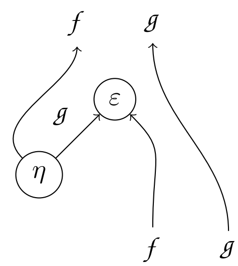
Exercise 1. Prove that for any 2-category $\mathbf{C}$ and any adjunction $f \dashv g$ in $\mathbf{C}$, one can build a monad in $\mathbf{C}$ whose underlying morphism is $f\,;g$.

Proof. Let f be from $\mathcal{C}$ to $\mathcal{C}$, and let η and ε be the unit and counit of the adjunction. Then $\langle \mathcal{C}, f\,;g, \mu, \mathfrak{a}, \eta, \mathfrak{i}\rangle$ is an adjunction, where μ, $\mathfrak{d}$, and $\mathfrak{i}$ are defined as follows:

$\mu : f\,;g\,;f\,;g \Rightarrow f\,;g =$ 

$\mathfrak{a}$ is given by the fact that both compositions result in the following string diagram: 

$\mathfrak{i}$ is given by 

and 

equal 

due to adjunction properties .

□

Exercise 2. Prove that, in the 2-category $\mathbf{CAT}$, for every monad $\mathcal{M}$ with underlying functor M on a category $\mathbf{C}$ there is some adjunction $F \dashv U$ such that M equals $F\,;U$. Hint: use the underlying functor $U : \mathbf{Alg}(\mathcal{M}) \to \mathbf{C}$ as the right adjoint.

Proof. Let $\mathcal{M}$ be $\langle \mathbf{C}, M, \mu, \mathfrak{d}, \eta, \mathfrak{i}\rangle$. Let $U : \mathbf{Alg}(\mathcal{M}) \to \mathbf{C}$ be the underlying functor of $\mathbf{Alg}(\mathcal{M})$. Let $F : \mathbf{C} \to \mathbf{Alg}(\mathcal{M})$ be the functor mapping each object $\mathcal{C}$ to the algebra $\langle M(\mathcal{C}), \mu_{\mathcal{C}}, \mathfrak{a}, \mathfrak{i}\rangle$ and each morphism $f : \mathcal{C}_1 \to \mathcal{C}_2$ to the algebra morphism $\langle M(f), \mathfrak{d}_f\rangle$ where $\mathfrak{d}_f : M(M(f))\,; \mu_{\mathcal{C}_2} = \mu_{\mathcal{C}_1}\,; M(f)$ comes from naturality of μ. The fact that F is functorial comes from functoriality of M. $F\,;U$ then equals M, so we can define the unit of the adjunction $\eta : \mathbf{C} \Rightarrow F\,;U$ as the unit of the monad $\eta : \mathbf{C} \Rightarrow M$. For the counit ε, we map each algebra $\langle \mathcal{C}, a, \mathfrak{d}_a, \bullet\rangle$ to the morphism of algebras $\langle a, \mathfrak{d}_a\rangle : U(F(\langle \mathcal{C}, a, \mathfrak{d}_a, \bullet\rangle)) = \langle M(\mathcal{C}), \mu_{\mathcal{C}}, \mathfrak{d}, \mathfrak{i}\rangle \to \langle \mathcal{C}, a, \mathfrak{d}_a, \bullet\rangle$. ε is natural because all algebra morphisms are distributive. Lastly, $((\eta \cdot F)\,;(F \cdot \varepsilon))_{\mathcal{C}}$ is defined as $M(\eta_{\mathcal{C}})\,; \mu_{\mathcal{C}}$ which equals the identity since η is an identity of μ, and $((U \cdot \eta)\,;(\varepsilon \cdot U))_{\langle \mathcal{C}, a, \bullet, \mathfrak{i}_a\rangle}$ is defined as $\eta_{\mathcal{C}}\,; a$ which equals the identity by $\mathfrak{i}_a$. Thus, $\langle \mathbf{C}, F, U, \eta, \varepsilon, \bullet, \bullet\rangle$ forms an adjunction with $F\,;U$ equal to M. □