Existentials

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November 3, 2014

Exercise 1. Prove that the existential subtyping relation for a category **Bnd** and functor **Bnd** $\to$ **Prost** is always a preorder. Be detailed in your proof.

Proof. Given an existential type $\exists\Gamma.\ \tau$, we can prove reflexivity using the identity morphism $id_\Gamma : \Gamma \to \Gamma$, since $\tau \leq \tau[id_\Gamma]$ holds due to the fact that functors preserve identities, so $\tau[id_\Gamma]$ must equal τ, and $\leq$ is assumed to be reflexive.

For transitivity, suppose we know $\exists\Gamma_1.\ \tau_1 \leq \exists\Gamma_2.\ \tau_2 \leq \exists\Gamma_3.\ \tau_3$ holds. This is possible only if there exist $\theta : \Gamma_2 \to \Gamma_1$ and $\theta' : \Gamma_3 \to \Gamma_2$ such that $\tau_1 \leq \tau_2[\theta]$ and $\tau_2 \leq \tau_3[\theta']$ hold. We demonstrate that $\exists\Gamma_1.\ \tau_1 \leq \exists\Gamma_3.\ \tau_3$ holds by using the morphism $\theta' \,;\theta : \Gamma_3 \to \Gamma_1$. $\tau_1 \leq \tau_2[\theta]$ holds by assumption; $\tau_2[\theta] \leq \tau_3[\theta'][\theta]$ holds because $\tau_2 \leq \tau_3[\theta']$ holds by assumption and $[\theta]$ is a morphism of **Prost** and therefore preserves subtypings; and $\tau_3[\theta'][\theta]$ equals $\tau_3[\theta' \,;\theta]$ because functors are distributive. So, $\tau_1 \leq \tau_3[\theta' \,;\theta]$ holds due to the assumed transitivity of $\leq$. This proves that $\exists\Gamma_1.\ \tau_1$ is a subtype of $\exists\Gamma_3.\ \tau_3$. □

Exercise 2. Prove that **Prost** has an $(\mathcal{E}, \mathcal{M})$ factorization structure, where $\mathcal{E}$ is the set of all epimorphisms and $\mathcal{M}$ is the set of all *initial* mono-sources. A **Prost**-source $(\mathcal{C} \xrightarrow{f_i} \mathcal{C}_i)_{i:I}$ is initial when for all pairs $c_1, c_2 : C$, if $\forall i : I.\ f_i(c_1) \leq f_i(c_2)$ holds then $c_1 \leq c_2$ holds as well (note that the reverse implication always holds because each function must be relation-preserving). You may assume that **Set** has an epi-mono factorization structure, that a morphism in **Prost** is an epimorphism iff its underlying function is an epimorphism, and that a source in **Prost** is a mono-source iff its underlying source is a mono-source.

Proof. First, we demonstrate that **Prost** has epi-initial-mono factorizations. Given a source $(f_i : \mathcal{R} \to \mathcal{R}_i)_{i \in I}$ let $\langle e : R \to X, (m_i : X \to R_i)_{i \in I} \rangle$ be the epi-mono factorization in **Set**. Then, define $x \sqsubseteq x'$ to be $\forall i.\ m_i(x) \leq m_i(x')$ (which is a preorder because each $\mathcal{R}_i$ defines a preorder). This makes the source $(m_i : \langle X, \sqsubseteq \rangle \to \mathcal{R}_i)$ initial by definition (and clearly each m_i is relation-preserving); it is also mono because its underlying source is mono. Next, suppose $r \leq r'$ holds in $\mathcal{R}$, then for any $i \in I$ we know $m_i(e(r)) \leq m_i(e(r'))$ holds, so $e(r) \sqsubseteq e(r')$ holds by definition of $\sqsubseteq$. Thus, $e : \mathcal{R} \to \langle X, \sqsubseteq \rangle$ is a morphism in **Prost**, and it is epi since its underlying function is epi. This makes $\langle e, (m_i)_{i \in I} \rangle$ an epi-initial-mono factorization of $(f_i)_{i \in I}$.

Second, we demonstrate that **Prost** has unique epi-initial-mono diagonalizations. Suppose we have that $e \,; g_i$ equals $f \,; m_i$ in **Prost** for all $i \in I$, where e is epi and $(m_i)_{i \in I}$ is initial mono. That implies we also have that $e \,; g_i$ equals $f \,; m_i$ in **Set** for all $i \in I$, where e is epi and $(m_i)_{i \in I}$ is mono. Because **Set** has unique epi-mono diagonalizations, there exists a unique function d such that $e \,; d$ equals f and $d \,; m_i$ equals g_i for all $i \in I$. Thus, the only thing we need to do is prove that d is relation-preserving. Suppose we have that $x \leq x'$ holds in the codomain of e. Then $m_i(d(x)) \leq m_i(d(x'))$ holds for all $i \in I$ because the left equals $f_i(x)$ and the right equals $f_i(x')$ and each f_i is relation-preserving. Because $(m_i)_{i \in I}$ is initial, this implies $d(x) \leq d(x')$ also holds. Thus, there exists a unique morphism d in **Prost** such that $e \,; d$ equals f and $d \,; m_i$ equals g_i for all $i \in I$. □