Effects

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Exercise 1. Prove that the functor $I : C \to \text{Eff}(M)$ for any monad $M$ on a category $C$ has a right adjoint $R$ such that the process for building a monad out of the adjunction $I \dashv R$ results in $M$. ($I$ is given in Exercise 2 of the Kliesi Categories lecture notes. You may assume $I$ is distributive and preserves identities.)

To maintain sanity, use $id$ and $\cdot$ for identity and composition in $C$, and use $id^*$ and $\cdot^*$ for identity and composition in $\text{Eff}(M)$. Similarly, if $f : C_1 \to M(C_2)$ is a morphism in $C$, then $f^* : C_1 \to C_2$ is the corresponding morphism in $\text{Eff}(M)$.

Proof. Let $R$ map $C$ to $M(C)$ and $f^* : C_1 \to C_2$ to $M(f) : \mu_{C_2}$. The identity of $C$ gets mapped to $M(id^*) : \mu_C$, which equals the identity since $id^* = \eta$ is an identity of $\mu$. The composition $f^* \cdot g^*$ gets mapped to $M(f \cdot M(g)) : \mu$, which equals $M(f) \cdot M(g) \cdot \mu$ by distributivity of $M$, which equals $M(f) \cdot M(g) : \mu$ by associativity of $\mu$, which equals $M(f) : \mu$. $M(g) : \mu$ by naturality of $\mu$, which is the composition of what $f^*$ and $g^*$ each get mapped to. Thus, $R$ is a functor.

$I$ maps the object $C$ to $C$, which $R$ maps to $M(C)$. $I$ maps the morphism $f$ to $(f ; \eta)^*$, which $R$ maps to $M(f ; \eta) : \mu$, which equals $M(f) : \mu$, which equals $M(f)$ because $\eta$ is an identity of $\mu$. Thus, $I : R$ equals $M$.

The unit of the adjunction is simply the unit of the monad. The counit $\varepsilon : R \circ I \Rightarrow \text{Eff}(M)$ is the natural transformation mapping each object $C$ to the morphism $(id_{M(C)})^* : I(R(C)) = M(C) \Rightarrow C$, which is natural because $I(R(f^*)) \cdot \varepsilon_{C_2} = (M(f) : \mu_{C_2} : \eta_{M(C_2)})^* \cdot (id_{M(C_2)})^* = (M(f) : \mu_{C_2} : \eta_{M(C_2)} : M(id_{M(C_2)}) : \mu_{C_2})^*$ which equals $(id_{M(C_2)} : M(f) : \mu_{C_2})^*$, the definition of $(\varepsilon_{C_2})^* : f^*$, since $M$ preserves identities and $\eta$ is an identity of $\mu$.

Finally, $I(\eta_C) \cdot \varepsilon_{I(C)}$ is defined as $(\eta_C : \eta_{M(C)} : M(id_{M(C)}) : \mu_C)^*$, which equals $(\eta_C : \eta_{M(C)} : M(id_{M(C)}) : \mu_C)^*$, which equals $(\eta_C : \eta_{M(C)} : M(id_{M(C)}) : \mu_C)^*$, which equals $\eta_{M(C)} : M(id_{M(C)}) : \mu_C$, which equals $id_{R(C)}$ because $M$ preserves identities and $\eta$ is an identity of $\mu$.

Exercise 2. Preordered monoids are the internal monoids of $\text{Prost}$. Prove that there is a function from the set of preordered monoids to the set of effectoids such that the set of the effects of an output of this function is the underlying set of the corresponding input.

Proof. Let $\langle E, \varepsilon, *, \leq, \cdot \rangle$ be a preordered monoid. Define $\varepsilon \mapsto \varepsilon$ to be $\varepsilon \leq \varepsilon$. Define $\varepsilon_1 \cdot \varepsilon_2 \mapsto \varepsilon$ to be $\varepsilon_1 \mapsto \varepsilon_2 \leq \varepsilon$. This, along with $\leq$, defines an effectoid with the effect set $E$; all we have to do is prove the required properties.

Identity Suppose we have $\varepsilon, \varepsilon' : E$. If there exists $\varepsilon_1$ such that $\varepsilon_1 \leq \varepsilon$ and $\varepsilon_1 \cdot \varepsilon \leq \varepsilon'$, then due to identity and congruence and exploiting transitivity we have $\varepsilon_1 \cdot \varepsilon_1 \leq \varepsilon_1 \cdot \varepsilon \leq \varepsilon'$, which equals $\varepsilon \leq \varepsilon'$, then if we define $\varepsilon_1$ and $\varepsilon_2$ to be $\varepsilon$ we know that $\varepsilon_1 \leq \varepsilon_2$ by reflexivity and $\varepsilon_1 \cdot \varepsilon \leq \varepsilon \cdot \varepsilon$ by identity. Similar arguments apply for right identity.

Associtivity Suppose we have $\varepsilon_1, \varepsilon_2, \varepsilon_3, \cdot : E$. If there exists some $\varepsilon_1$ such that $\varepsilon_1 \cdot \varepsilon_2 \leq \varepsilon$ and $\varepsilon_1 \cdot \varepsilon_3 \leq \varepsilon$, then we can define $\varepsilon_1 \cdot \varepsilon_2 \leq \varepsilon$ so that $\varepsilon_2 \cdot \varepsilon_3 \leq \varepsilon$ holds by reflexivity and, exploiting transitivity, $\varepsilon_1 \cdot \varepsilon_2 \leq \varepsilon_1 \cdot \varepsilon_3 \leq \varepsilon$ holds due to associativity and congruence. A similar argument holds for the reverse implication.

Reflexivity Holds by definition of preorder.

Congruence Holds due to transitivity and the definition of $\varepsilon \mapsto \varepsilon$ and $\varepsilon_1 \cdot \varepsilon_2 \mapsto \varepsilon$.

Exercise 3. Suppose that we want to build a productoid for the effectoid arising from the preorder monoid $P\{\{1,2\}\}_{\leq, \cup}$. Because this monoid is idempotent, it turns out any such productoid would provide a monadic structure (i.e. unit and join) for each $m_r$. Suppose we want to require the monad for $m_{\emptyset}$ to be the identity monad (meaning the identity 1-cell with identity 2-cells for unit and join). Suppose furthermore we want to require $m_{\{1,2\}}$ to equal $m_{\{1\}} \cdot m_{\{2\}}$ and require $\mu_{\{1\}} \cup_{\{2\}}$ to be the identity 2-cell of $m_{\{1\}} \cdot m_{\{2\}}$. Let $m_1$ denote $m_{\{1\}}$ and $m_2$ denote $m_{\{2\}}$. It turns out that building such a productoid would require just one more 2-cell $\delta : m_2 ; m_1 \Rightarrow m_1 ; m_2$ satisfying four equations (without needing to use universal quantifiers). Determine what these four equations are, though do not provide the proof that they are necessary and sufficient to build a productoid satisfying the required property. Hint: $\delta$ corresponds to $\mu_{\{2\}} \cup_{\{1\}}$. Also, save time by copy-pasting the diagrams from the lecture notes.
Proof. Let \( \eta_1 \) and \( \mu_1 \) denote the unit and join for \( m_1 \), and let \( \eta_2 \) and \( \mu_2 \) denote the unit and join for \( m_2 \).