Exercise 1. Prove that the inclusion functor $\text{Set} \hookrightarrow \text{Rel}$ has a right adjoint. You may use any of the equivalent definitions of adjunction. For clarification, $I$ is the functor mapping each set $X$ (an object of $\text{Set}$) to the set $X$ (also an object of $\text{Rel}$) and each function $X \to Y$ (a morphism of $\text{Set}$) to the relation $\lambda(x,y)$. $f(x) = y$ (a morphism of $\text{Rel}$).

Proof. There is a functor $P : \text{Rel} \to \text{Set}$ mapping each set $X$ to the set $PX$ and each relation $R : X \times Y \to \text{Prop}$ to the function $\lambda x. \{y : Y \mid \exists x \in X. x R y\}$. The identity relation $\lambda(x_1,x_2).x_1 = x_2$ gets mapped to the function $\lambda x. \{x_2 : X \mid \exists x_1 \in x. x_1 = x_2\}$ which is simply the identity function. The composition relation $\lambda(x,z)$. $\exists y : Y. x R_1 y \land y R_2 z$ gets mapped to the function $\lambda x. \{z : Z \mid \exists x \in X. \exists y : Y. x R_1 y \land y R_2 z\}$ which equals $\lambda x. \{z : Z \mid \exists y \in \{y : Y \mid \exists x \in X. x R_1 y\}. y R_2 z\}$, proving distributivity.

Given a $\text{Rel}$-object $Y$, define the $\text{Rel}$-morphism $\varepsilon_Y : I(P(Y)) \to Y$ to be the binary relation $\lambda(y,z). y \in \bar{y}$. Given another $\text{Rel}$-morphism from some $I(X)$ to $Y$, i.e. a binary relation $R : X \times Y \to \text{Prop}$, the unique corresponding set-morphism from $X$ to $P(Y)$ is the function $\lambda x. \{y : Y \mid x R y\}$. The $\text{Rel}$-composition $I(\lambda x. \{y : Y \mid x R y\}).(\lambda(y,z). y \in \bar{y})$ is by definition the binary relation $\lambda(x,y). \exists \bar{y} : P(Y). \{y : Y \mid x R y\} = \bar{y} \land y \in \bar{y}$, which is equivalent to simply $R$. Furthermore, for any function $f : X \to P(Y)$, the composition $\lambda(x,y). \exists \bar{y} : P(Y). f(x) = \bar{y} \land y \in \bar{y}$ is equivalent to $\lambda(x,y). y \in f(x)$, which is equivalent to $R$ if and only if $f(x) = \{y : Y \mid \exists x : X. x R y\}$, making the function corresponding to $R$ unique.

Exercise 2. There is a functor from $\textbf{1}$ to $\text{Set}$ picking out the empty set, and another functor from $\textbf{1}$ to $\text{Set}$ picking out the singleton set. One is the left adjoint to the unique functor from $\text{Set}$ to $\textbf{1}$, and the other is the right adjoint to the unique functor from $\text{Set}$ to $\textbf{1}$. Determine and prove which is which.

Proof. The functor $F : \textbf{1} \to \text{Set}$ picking out the empty set is the left adjoint to the unique functor $\emptyset$ from $\text{Set}$ to $\textbf{1}$ (whose only object we call $\ast$). For any $X : \text{Set}$ and $\ast : \textbf{1}$, both $M_{\text{Set}}(F(\ast),X)$ and $M_{\text{Set}}(\ast,\langle\emptyset\rangle(X))$ have only one element, making them isomorphic. Furthermore, since $\textbf{1}$ has only one morphism, this isomorphism is guaranteed to be natural, making this an adjunction.

The functor $G : \textbf{1} \to \text{Set}$ picking out the singleton set is the right adjoint to the unique functor $\emptyset$ from $\text{Set}$ to $\textbf{1}$ (whose only object we call $\ast$). For any $X : \text{Set}$ and $\ast : \textbf{1}$, both $M_{\text{Set}}(\langle\emptyset\rangle(X),\ast)$ and $M_{\text{Set}}(\langle\emptyset\rangle,\langle\emptyset\rangle(X))$ have only one element, making them isomorphic. Furthermore, since $\textbf{1}$ has only one morphism, this isomorphism is guaranteed to be natural, making this an adjunction.

Exercise 3. $N : \textbf{1} \to \text{Set}$ maps the only object of $\textbf{1}$ to the set $\mathbb{N}$. repeat is the natural transformation from $N$ to $N ; L$ (i.e. $L(N)$) mapping the sole object of $\textbf{1}$ to the function mapping $n$ to the length-$n$ list $[n,\ldots,n]$. sum is the natural transformation from $N ; L$ to $N$ mapping the sole object of $\textbf{1}$ to the function mapping a list of numbers and returns its sum.

The string diagram to the right denotes a natural transformation from the functor $N : \textbf{1} \to \text{Set}$ to itself ($N$ maps the only object of $\textbf{1}$ to the set $\mathbb{N}$). In particular, this means it describes a function from $\mathbb{N}$ to $\mathbb{N}$. Determine what that function is in terms of basic arithmetic. (No proof necessary; the purpose of this is to learn the notation.)

Proof. The function is $\lambda n. n^3$. The program described by the diagram is $\lambda n. \text{sum} (\text{flatten} (\text{map} \text{\text{\textit{repeat}}(\text{\textit{repeat}}(n))))).$