Adjunctions

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Exercise 1. Prove that the inclusion functor $\textbf{Set} \overset{I}{\hookrightarrow} \textbf{Rel}$ has a right adjoint. You may use any of the equivalent definitions of adjunction. For clarification, I is the functor mapping each set X (an object of **Set**) to the set X (also an object of **Rel**) and each function $X \to Y$ (a morphism of **Set**) to the relation $\lambda\langle x, y\rangle.\ f(x) = y$ (a morphism of **Rel**).

Proof. There is a functor $\mathbb{P} : \textbf{Rel} \to \textbf{Set}$ mapping each set X to the set $\mathbb{P}X$ and each relation $R : X \times Y \to \mathit{Prop}$ to the function $\lambda\vec{x}.\ \{y : Y \mid \exists x \in \vec{x}.\ x\ R\ y\}$. The identity relation $\lambda\langle x_1, x_2\rangle.x_1 = x_2$ gets mapped to the function $\lambda\vec{x}.\ \{x_2 : X \mid \exists x_1 \in \vec{x}.\ x_1 = x_2\}$ which is simply the identity function. The composition relation $\lambda\langle x, z\rangle.\ \exists y : Y.\ x\ R_1\ y \wedge y\ R_2\ z$ gets mapped to the function $\lambda\vec{x}.\ \{z : Z \mid \exists x \in \vec{x}.\ \exists y : Y.\ x\ R_1\ y \wedge y\ R_2\ z\}$ which equals $\lambda\vec{x}.\ \{z : Z \mid \exists y \in \{y : Y \mid \exists x \in \vec{x}.\ x\ R_1\ y\}.\ y\ R_2\ z\}$, proving distributivity.

Given a **Rel**-object Y, define the **Rel**-morphism $\varepsilon_Y : I(\mathbb{P}(Y)) \to Y$ to be the binary relation $\lambda\langle\vec{y}, y\rangle.\ y \in \vec{y}$. Given another **Rel**-morphism from some $I(X)$ to Y, i.e. a binary relation $R : X \times Y \to \texttt{Prop}$, the unique corresponding **Set**-morphism from X to $\mathbb{P}(Y)$ is the function $\lambda x.\ \{y : Y \mid x\ R\ y\}$. The **Rel**-composition $I(\lambda x.\ \{y : Y \mid x\ R\ y\})\,;(\lambda\langle\vec{y}, y\rangle.\ y \in \vec{y})$ is by definition the binary relation $\lambda\langle x, y\rangle.\ \exists\vec{y} : \mathbb{P}(Y).\ \{y : Y \mid x\ R\ y\} = \vec{y} \wedge y \in \vec{y}$, which is equivalent to simply R. Furthermore, for any function $f : X \to \mathbb{P}Y$, the composition $\lambda\langle x, y\rangle.\ \exists\vec{y} : \mathbb{P}Y.\ f(x) = \vec{y} \wedge y \in \vec{y}$ is equivalent to $\lambda\langle x, y\rangle.\ y \in f(x)$, which is equivalent to R if and only if $f(x) = \{y : Y \mid \exists x : X.\ x\ R\ y\}$, making the function corresponding to R unique. □

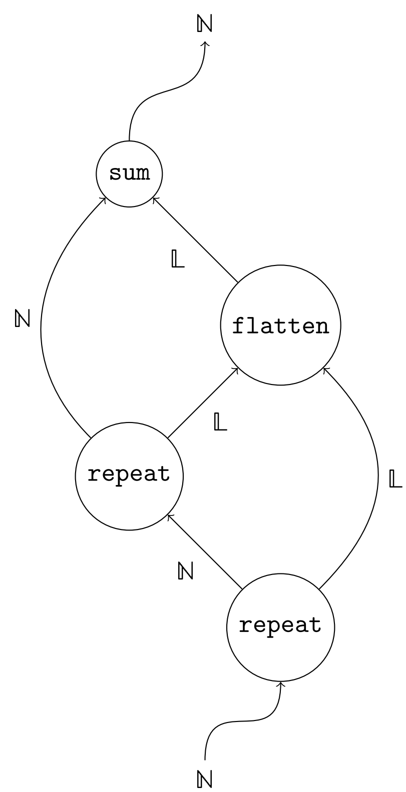
Exercise 2. There is a functor from **1** to **Set** picking out the empty set, and another functor from **1** to **Set** picking out the singleton set. One is the left adjoint to the unique functor from **Set** to **1**, and the other is the right adjoint to the unique functor from **Set** to **1**. Determine and prove which is which.

Proof. The functor $F : \mathbf{1} \to \textbf{Set}$ picking out the empty set is the left adjoint to the unique functor $\langle\rangle$ from **Set** to **1** (whose only object we call $\star$). For any $X : \textbf{Set}$ and $\star : \mathbf{1}$, both $M_{\textbf{Set}}(F(\star), X)$ and $M_{\mathbf{1}}(\star, \langle\rangle(X))$ have only one element, making them isomorphic. Furthermore, since **1** has only one morphism, this isomorphism is guaranteed to be natural, making this an adjunction.

The functor $G : \mathbf{1} \to \textbf{Set}$ picking out the singleton set is the right adjoint to the unique functor $\langle\rangle$ from **Set** to **1** (whose only object we call $\star$). For any $X : \textbf{Set}$ and $\star : \mathbf{1}$, both $M_{\mathbf{1}}(\langle\rangle(X), \star)$ and $M_{\textbf{Set}}(X, G(\star))$ have only one element, making them isomorphic. Furthermore, since **1** has only one morphism, this isomorphism is guaranteed to be natural, making this an adjunction. □

Exercise 3. $\mathbb{N} : \mathbf{1} \to \textbf{Set}$ maps the only object of **1** to the set $\mathbb{N}$. `repeat` is the natural transformation from $\mathbb{N}$ to $\mathbb{N}\,;\mathbb{L}$ (i.e. $\mathbb{L}(\mathbb{N})$) mapping the sole object of **1** to the function mapping n to the length-n list $[n, \ldots, n]$. `sum` is the natural transformation from $\mathbb{N}\,;\mathbb{L}$ to $\mathbb{N}$ mapping the sole object of **1** to the function mapping a list of numbers and returns its sum.

The string diagram to the right denotes a natural transformation from the functor $\mathbb{N} : \mathbf{1} \to \textbf{Set}$ to itself ($\mathbb{N}$ maps the only object of **1** to the set $\mathbb{N}$). In particular, this means it describes a function from $\mathbb{N}$ to $\mathbb{N}$. Determine what that function is in terms of basic arithmetic. (No proof necessary; the purpose of this is to learn the notation.)



Proof. The function is $\lambda n.\ n^3$. The program described by the diagram is $\lambda n.\ \texttt{sum}(\texttt{flatten}(\texttt{map}_{\texttt{repeat}}(\texttt{repeat}(n))))$. □