# Transpositions

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**Definition** ($G$-Structured Arrow for a functor $G : \mathbf{D} \to \mathbf{C}$ and an object $\mathcal{C} : \mathbf{C}$)**.** An object $\mathcal{D} : \mathbf{D}$ and a morphism $f : \mathcal{C} \to G(\mathcal{D})$. A morphism of $G$-structured arrows from $\mathcal{C} \xrightarrow{f_1} G(\mathcal{D}_1)$ to $\mathcal{C} \xrightarrow{f_2} G(\mathcal{D}_2)$ is a morphism $\mathcal{D}_1 \xrightarrow{d} \mathcal{D}_2$ such that $f_1 \,;G(d)$ equals $f_2$.

**Definition** ($F$-Costructured Arrow for a functor $F : \mathbf{C} \to \mathbf{D}$ and an object $\mathcal{D} : \mathbf{D}$)**.** An object $\mathcal{C} : \mathbf{C}$ and a morphism $g : F(\mathcal{C}) \to \mathcal{D}$. A morphism of $F$-structured arrows from $F(\mathcal{C}_1) \xrightarrow{g_1} \mathcal{D}$ to $F(\mathcal{C}_2) \xrightarrow{g_2} \mathcal{D}$ is a morphism $\mathcal{C}_1 \xrightarrow{c} \mathcal{C}_2$ such that $F(c)\,; g_2$ equals $g_1$.

**Definition** (Adjunction (via Universal (Co-)Structured Arrows))**.** A pair of functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ with either (the following two conditions are equivalent)

- for each object $\mathcal{C} : \mathbf{C}$ a morphism $\mathcal{C} \xrightarrow{\eta_{\mathcal{C}}} G(F(\mathcal{C}))$ with the property that for any object $\mathcal{D} : \mathbf{D}$ and morphism $f : \mathcal{C} \to G(\mathcal{D})$ there exists a unique morphism $f^{\leftarrow} : F(\mathcal{C}) \to \mathcal{D}$ such that $\eta_{\mathcal{C}}\,; G(f^{\leftarrow})$ equals $f$
- for each object $\mathcal{D} : \mathbf{D}$ a morphism $F(G(\mathcal{D})) \xrightarrow{\varepsilon_{\mathcal{D}}} \mathcal{D}$ with the property that for any object $\mathcal{C} : \mathbf{C}$ and morphism $g : F(\mathcal{C}) \to \mathcal{D}$ there exists a unique morphism $g^{\rightarrow} : \mathcal{C} \to G(\mathcal{D})$ such that $F(g^{\rightarrow})\,; \varepsilon_{\mathcal{D}}$ equals $g$

*Remark.* $\eta$ is called the unit. $\varepsilon$ is called the counit.

**Definition** (Adjunction (via Transposition))**.** A pair of functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ with a bijection $\forall \mathcal{C} : \mathbf{C}, \mathcal{D} : \mathbf{D}.\ (F\mathcal{C} \to \mathcal{D}) \underset{\bullet^{\leftarrow}}{\overset{\bullet^{\rightarrow}}{\rightleftarrows}} (\mathcal{C} \to G\mathcal{D})$ that is natural with respect to the quantified $\mathcal{C}$ and $\mathcal{D}$, meaning the following holds: $\forall F\mathcal{C}_2 \xrightarrow{g} \mathcal{D}_1 : \mathbf{D}, \mathcal{C}_1 \xrightarrow{c} \mathcal{C}_2 : \mathbf{C}, \mathcal{D}_1 \xrightarrow{d} \mathcal{D}_2.\ (F c\,; g\,; d)^{\rightarrow} = c\,; g^{\rightarrow}\,; G d$, or equivalently $\forall \mathcal{C}_2 \xrightarrow{f} G\mathcal{D}_1 : \mathbf{C}, \mathcal{C}_1 \xrightarrow{c} \mathcal{C}_2 : \mathbf{C}, \mathcal{D}_1 \xrightarrow{d} \mathcal{D}_2.\ (c\,; f\,; G d)^{\leftarrow} = F c\,; f^{\leftarrow}\,; d$.

**Exercise 1.** Prove that the above two definitions of adjunction are equivalent (i.e. there is a bijection between them).

**Definition** (Left/Right Adjoint)**.** Given an adjuntion with $F$ and $G$ as above, $F$ is called the left adjoint and $G$ is called the right adjoint. A functor is called a left/right adjoint if it is the left/right adjoint of some adjunction.

*Remark.* The reason $F$ is the left whereas $G$ is the right is that the isomoprhism is between arrows with $F$ applied to the domain (i.e. to the left of $\to$) and arrows with $G$ applied to the codomain (i.e. to the right of $\to$). We use $^{\rightarrow}$ because changes morphisms from the left form into the right form, and $^{\leftarrow}$ does the reverse.

**Exercise 2.** Suppose functors $F$ and $G$ have two ways to instantiate $\eta$, $\varepsilon$, or the isomorphism. Prove that these two instantiations must be isomoprhic to each other according to the appropriate notion of isomorphism.

*Notation.* $F \dashv G$ means that $F$ and $G$ are the left and right adjoints of some adjunction.

**Example.** A subcategory $\mathbf{S} \overset{I}{\hookrightarrow} \mathbf{C}$ is reflective precisely when $I$ is a right adjoint. The left adjoint is $R$. The unit is the reflection arrows.

**Example.** The functor $F : \mathbf{Set} \to \mathbf{Mon}$ mapping a set $X$ to $(\mathbb{L}X)_{++}$ is left adjoint to the underlying functor $U : \mathbf{Mon} \to \mathbf{Set}$.

**Example.** The functor $F : \mathbf{Set} \to \mathbf{Alg}(2, 0)$ mapping a set $X$ to the algebra of expressions with a binary operation, a nullary operation, and all free variables in $X$, and mapping functions $f$ to the algebra homomorphism simply using $f$ to rename variables in expressions, is left adjoint to the underlying functor $U : \mathbf{Alg}(2, 0) \to \mathbf{Set}$. If $\theta$ is a function from $X$ to elements of some algebra, then $f^{\leftarrow}$ is the algebra homomorphism mapping expressions to their evaluation in that algebra using the valuation $\theta$ for variables.

*Remark.* In general, a left adjoint to an underlying functor is called a free functor. Consequently, $(\mathbb{L}X)_{++}$ is called the free monoid of $X$.

**Exercise 3.** Show that the inclusion functor $\mathbf{Set} \hookrightarrow \mathbf{Rel}$ has a right adjoint. This means $\mathbf{Set}$ is a *coreflective* subcategory of $\mathbf{Rel}$.