Natural Transformations

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Definition (Natural Transformation from $F : \mathbf{C} \to \mathbf{D}$ to $G : \mathbf{C} \to \mathbf{D}$)**.** A tuple $\langle \alpha, \mathfrak{n} \rangle$ where the components have the following types:

> **Transformation** α**:** For every object $\mathcal{C}$ of $\mathbf{C}$, a morphism $\alpha_{\mathcal{C}} : F(\mathcal{C}) \to G(\mathcal{C})$ of $\mathbf{D}$
>
> **Naturality** $\mathfrak{n}$**:** $\forall \mathcal{C}_1 \xrightarrow{m} \mathcal{C}_2 : \mathbf{C}.\ \alpha_{\mathcal{C}_1} ; G(m) = F(m) ; \alpha_{\mathcal{C}_2}$

Notation. $F \Rightarrow G$ denotes the set of natural transformations from F to G.

Notation. The identity morphism on an object $\mathcal{O}$ is often denoted simply as $\mathcal{O}$. Similarly, the identity functor on a category $\mathbf{C}$ is often denoted simply as $\mathbf{C}$.

Notation. The functor from $\mathbf{C}$ to $\mathbf{D}$ mapping everything to an object $\mathcal{D}$ or its identity morphism is denoted with $\mathcal{D}$.

Definition (Endofunctor)**.** A functor whose domain and codomain are the same.

Example. The following are natural transformations between endofunctors on **Set**:

`singleton` $: \mathbf{Set} \Rightarrow \mathbb{L}$**:** $\langle \lambda\tau.\ \lambda t.\ [t], \centerdot \rangle$

`doubleton` $: \mathbf{Set} \Rightarrow \mathbb{L}$**:** $\langle \lambda\tau.\ \lambda t.\ [t, t], \centerdot \rangle$

`flatten` $: \mathbb{L} ; \mathbb{L} \Rightarrow \mathbb{L}$**:** $\langle \lambda\tau.\ \lambda[\vec{t}_1, \ldots, \vec{t}_n].\ \vec{t}_1 +\!\!+ \ldots +\!\!+ \vec{t}_n, \centerdot \rangle$

`reverse` $: \mathbb{L} \Rightarrow \mathbb{L}$**:** $\langle \lambda\tau.\ \lambda[t_1, \ldots, t_n].\ [t_n, \ldots, t_1], \centerdot \rangle$

`length` $: \mathbb{L} \Rightarrow \mathbb{N}$**:** $\langle \lambda\tau.\ \lambda[t_1, \ldots, t_n].\ n, \centerdot \rangle$

Exercise 1. Prove that the set of natural transformations from $\mathcal{C}_1 : \mathbf{1} \to \mathbf{C}$ to $\mathcal{C}_2 : \mathbf{1} \to \mathbf{C}$ is isomorphic to the set of morphisms from the object selected by $\mathcal{C}_1$ to the object selected by $\mathcal{C}_2$.

Exercise 2. Prove that a natural transformation from $m_1 : \mathbf{2} \to \mathbf{C}$ to $m_2 : \mathbf{2} \to \mathbf{C}$ is a commuting square with the morphism selected by m_1 on the left and the morphism selected by m_2 on the right.

Exercise 3. Prove that the reflection arrows of a reflective subcategory $\mathbf{S} \overset{I}{\hookrightarrow} \mathbf{C}$ form a natural transformation $r : \mathbf{C} \Rightarrow R ; I : \mathbf{C} \to \mathbf{C}$.

Exercise 4. Prove that for any reflective subcategory there is a natural transformation $\varepsilon : I ; R \Rightarrow \mathbf{S}$. Specify this natural transformation in detail for the reflective subcategory $\mathbf{Mon} \hookrightarrow \mathbf{Sgr}$.

Exercise 5. Prove that a natural transformation could equivalently be defined as a tuple $\langle \alpha, \mathfrak{n} \rangle$ where the components have the following types:

> **Transformation** α**:** $\forall \mathcal{C}_1 \xrightarrow{m} \mathcal{C}_2 : \mathbf{C}.\ F(\mathcal{C}_1) \to G(\mathcal{C}_2)$
>
> **Naturality** $\mathfrak{n}$**:** $\forall \mathcal{C}_1 \xrightarrow{m_1} \mathcal{C}_2 \xrightarrow{m_2} \mathcal{C}_3.\ F(m_1) ; \alpha_{m_2} = \alpha_{m_1 ; m_2} = \alpha_{m_1} ; G(m_2)$

Exercise 6. Define the category $\mathbf{C} \twoheadrightarrow \mathbf{D}$ whose objects are functors from $\mathbf{C}$ to $\mathbf{D}$ and whose morphisms are natural transformations between those functors.

Exercise 7. Prove that `reverse` is an isomorphism in $\mathbf{Set} \twoheadrightarrow \mathbf{Set}$.

Exercise 8. Prove that the isomorphisms in $\mathbf{C} \multimap \mathbf{D}$ are precisely the natural transformations for which every morphism component is an isomorphism (referred to as a natural isomorphism).

Exercise 9. Prove that the monomorphisms in $\mathbf{C} \multimap \mathbf{D}$ are precisely the natural transformations for which every morphism component is a monomorphism (referred to as a natural monomorphism).

Exercise 10. Prove that there is a binary functor from $[\mathbf{C} \multimap \mathbf{D}, \mathbf{D} \multimap \mathbf{E}]$ to $\mathbf{C} \multimap \mathbf{E}$.

Exercise 11. Determine what 5 would be mapped to by $\texttt{singleton} \,;\, \texttt{doubleton} : \mathbf{Set} \Rightarrow \mathbb{L} \,;\, \mathbb{L}$.

Exercise 12. Prove that, for any reflective subcategory, $(r \,;\, R) \,;\! (R \,;\, \varepsilon) : R \Rightarrow R : \mathbf{C} \to \mathbf{S}$ and $(I \,;\, r) \,;\! (\varepsilon \,;\, I) : I \Rightarrow I : \mathbf{S} \to \mathbf{C}$ equal the identity natural transformation on R and I respectively, where we overload R and I to denote their identity natural transformations.