Topoi
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Definition (Subobject Classifier for a Category $\mathbf{C}$). An object $\Omega$ and a morphism $\textbf{true} : \top \rightarrow \Omega$ with the property that, for every monomorphism $m : S \hookrightarrow C$ in $\mathbf{C}$, there exists a unique morphism $\chi_m : C \rightarrow \Omega$, called the characteristic morphism of $m$, with the property that the following is a pullback square:

$$
\begin{array}{ccc}
S & \xrightarrow{m} & C \\
\downarrow \ & \ & \downarrow \chi_m \\
\top & \xrightarrow{\textbf{true}} & \Omega
\end{array}
$$

Example. $\mathbf{B}$ with $\textbf{true}$ is the subobject classifier for $\mathbf{Set}$. Given an injection $m : S \rightarrow C$, then $\chi_m$ is the function $\lambda c. \exists s. m(s) = c$.

Definition (Topos). A finitely complete category with exponentials (with respect to products, denoted $\rightarrow$) and a subobject classifier.

Example. The category $\mathbf{Set}$ and its full subcategory $\mathbf{Fin}$ of finite sets are both topoi.

Remark. Every morphism from $\top$ is a monomorphism (in any category with a terminal object, not just in topoi).

Theorem. One can implement $\land : \Omega \land \Omega \rightarrow \Omega$ as the characteristic morphism of $(\textbf{true}, \textbf{true}) : \top \rightarrow \Omega \land \Omega$. One can implement $\Rightarrow : \Omega \land \Omega \rightarrow \Omega$ as the characteristic morphism of the equalizer of $\pi_1$ and $\land$ from $\Omega \land \Omega$ to $\Omega$ (which works because $\phi \Rightarrow \psi$ holds if and only if $\phi \leftrightarrow \phi \land \psi$ holds).

Notation. Given a morphism $f : \mathbf{A} \& \mathbf{B} \rightarrow \mathbf{C}$, we denote the corresponding morphism from $\mathbf{B}$ to $\mathbf{A} \rightarrow \mathbf{C}$ with $\lambda A f$.

Theorem. Given an object $\mathbf{C}$, one can implement $\forall C : (\mathbf{C} \rightarrow \Omega) \rightarrow \Omega$ as the characteristic morphism for $\lambda C(\pi_2; \textbf{true}) : \top \rightarrow (\mathbf{C} \rightarrow \Omega)$.

Theorem. One can implement $\textbf{false} : \top \rightarrow \Omega$ as the morphism $(\lambda_{\Omega_1} \pi_1) ; \forall \Omega$ (which represents the proposition $\forall \phi : \text{Prop.} \phi$).

Theorem. The pullback of $\textbf{true} : \top \rightarrow \Omega$ and $\textbf{false} : \top \rightarrow \Omega$ is an initial object.

Theorem. One can use the above components to implement $\lor : \Omega \& \Omega \rightarrow \Omega$ via the predicate $\forall p : \Omega. (\phi \Rightarrow p) \land (\psi \Rightarrow p) \Rightarrow p$. Similarly, one can implement $\exists C : (\mathbf{C} \rightarrow \Omega) \rightarrow \Omega$ via the predicate $\forall p : \Omega. (\forall C. \phi(c) \Rightarrow p) \Rightarrow p$.

Theorem. One can implement $\equiv C : \mathbf{C} \& \mathbf{C} \rightarrow \Omega$ as the characteristic morphism of $(\text{id}_C, \text{id}_C) : \mathbf{C} \rightarrow \mathbf{C} \& \mathbf{C}$.

Definition (Natural-Numbers Object of a Category $\mathbf{C}$). An object $\mathcal{N}$ along with morphisms $e : \top \rightarrow \mathcal{N}$ and $s : \mathcal{N} \rightarrow \mathcal{N}$ with the property that, for every object $\mathbf{C}$ and morphisms $e_z : \mathbf{C} \rightarrow \mathbf{C}$ and $s_z : \mathbf{C} \rightarrow \mathbf{C}$, there exists a unique morphism $\text{ind}(e_z, s_z) : \mathcal{N} \rightarrow \mathbf{C}$ such that the following commutes:

$$
\begin{array}{ccc}
e_z & \xrightarrow{s} & \mathcal{N} \\
\top & \xrightarrow{\text{ind}(e_z, s_z)} & \mathbf{C}
\end{array}
$$

Example. $\mathbb{N}$ with 0 and $\lambda n. n + 1$ is a natural-numbers object of $\mathbf{Set}$. $\mathbf{Fin}$ has no natural-numbers object.

Theorem. All topoi are finitely cocomplete.

Definition (Boolean Topos). A topos with the property that $\top \xrightarrow{\textbf{true}} \Omega \xleftarrow{\textbf{false}} \top$ is a coproduct.

Definition (Two-Value Topos). A topos with exactly two morphisms from $\top$ to $\Omega$ (necessarily $\textbf{true}$ and $\textbf{false}$).

Definition (Well-Pointed). The property that for all $f, g : C_1 \rightarrow C_2$, $\forall e : \top \rightarrow C_1. e : f = e : g$ implies $f$ equals $g$.

Definition (Topos admitting the Axiom of Choice). A topos with the property that all epimorphisms are sections.

Theorem. Every topos admitting the axiom of choice is Boolean. Every well-pointed topos is two-value. Every well-pointed topos is Boolean (using a classical metatheory).