Subobjects

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Definition (Monomorphism)**.** A morphism $\mathcal{C}_1 \xrightarrow{m} \mathcal{C}_2$ such that for all $\mathcal{C} \underset{m_2}{\overset{m_1}{\rightrightarrows}} \mathcal{C}_1$, if $m_1\,;m$ equals $m_2\,;m$ then m_1 equals m_2.

Notation. Monomorphisms are indicated by $\hookrightarrow$.

Exercise 1. Prove that the monomorphisms in **Set**, **Prost**, $\mathbb{\Delta}_n$, $\mathbb{\Delta}$, and **Mon** are the injective functions.

Exercise 2. Classify the monomorphims in **Rel**. Hint: while left-total left-unique relations (i.e. relations corresponding to injective functions) are monomorphisms in **Rel**, not all monomorphisms in **Rel** fall under that classification.

Exercise 3. Prove that monomorphisms in **Mat** are the matrices where one can remove some select rows to obtain an invertible matrix.

Exercise 4. Prove that monomorphisms in **CAT** are precisely the functors whose component on objects and components on morphisms are all injective functions.

Exercise 5. Prove that all sections are monomorphisms, but not all monomorphisms are sections. Hint: use **Prost** for your counterexample.

Exercise 6. Prove that all retracts that are monomorphisms are also isomorphisms.

Remark. The above generalizes the proof that a surjection that is injective is a bijection.

Definition (Subobject of $\mathcal{C}$)**.** An object $\mathcal{S}$ and a monomorphism $m : \mathcal{S} \hookrightarrow \mathcal{C}$.

Remark. Often the monomorphism can be inferred from context, so we will say $\mathbb{N}_+$ is a subobject of $\mathbb{Z}_+$, even though technically we should say $\langle \mathbb{N}_+, \langle \lambda n.\ n \text{ as } \mathbb{Z}, \centerdot, \centerdot \rangle\rangle$ is a subobject of $\mathbb{Z}_+$, since after all $\langle \mathbb{N}_+, \langle \lambda n.\ -(n \text{ as } \mathbb{Z}), \centerdot, \centerdot \rangle\rangle$ is also a subobject of $\mathbb{Z}_+$.

Definition ($\mathbf{Sub}(\mathcal{C})$)**.** The category whose

Objects are the subobjects of $\mathcal{C}$

Morphisms $\langle \mathcal{S}_1, m_1\rangle \to \langle \mathcal{S}_2, m_2\rangle$ are morphisms $m : \mathcal{S}_1 \to \mathcal{S}_2$ such that $m_1 = m\,; m_2$

Composition is as in $\mathcal{C}$

Identity is as in $\mathcal{C}$

Definition (Subcategory)**.** A subcategory of a category $\mathcal{C}$ is a subobject in **CAT**.

Example. $\mathbf{Set} \hookrightarrow \mathbf{Rel}$

$\mathbb{P} : \mathbf{Rel} \hookrightarrow \mathbf{Set}$

$\mathbf{n} \hookrightarrow \omega$

$\mathbb{\Delta} \hookrightarrow \mathbf{Prost}$

$\mathbf{Mono} \hookrightarrow \mathbf{Alg}(2, 0)$

$\mathbf{Prost} \hookrightarrow \mathbf{Rel}(2)$

Definition (Wide Subcategory)**.** A subcategory whose inclusion functor is surjective on objects (i.e. F_O is a surjection).

Definition (Full Subcategory)**.** A subcategory whose inclusion functor is surjective on morphisms for each object pair (i.e. $F_M(\mathcal{C}_1, \mathcal{C}_2)$ is a surjection for all objects $\mathcal{C}_1$ and $\mathcal{C}_2$).

Exercise 7. Prove that **Mon** is isomorphic in **CAT** to the full subcategory of **Cat** containing only the categories with one object.