**Productors**

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**Definition** (Productor of an Effector \(\langle E, \stackrel{i}{\to}, \cdot, \cdot \rangle\) for a 2-Category \(C\)). A tuple \(\langle C, m, \mu, a, i \rangle\) whose components have the following types:

**Object** \(C\) is an object of \(C\)

**Morphisms** \(m\) maps each \(\epsilon : E\) to a morphism \(m_\epsilon\) of \(C\) from \(C\) to \(C\)

**Join** \(\mu\) maps each related pair \([\epsilon_1, \ldots, \epsilon_n]\) to \(\epsilon\) to a 2-cell \(\mu_{[\epsilon_1, \ldots, \epsilon_n]}\) of \(C\) from \(m_{\epsilon_1} ; \ldots ; m_{\epsilon_n} \Rightarrow m_\epsilon\)

**Associativity** \(a\) proves

In other words,

**Identity** \(i\) is a proof that \(\forall \epsilon : E. \mu_\epsilon = \text{id}_{m_\epsilon} : m_\epsilon \Rightarrow m_\epsilon\)

**Exercise 1.** Prove that a monad is a productor for the effector with one element and with \(\stackrel{i}{\to}\) always true.

**Exercise 2.** Every effector corresponds to a thin multicategory, which in turn corresponds to an opetory with 1 object and at most one 2-cell from any domain to any codomain. Prove that a productor for an effector is simply a functor from the corresponding opetory.

**Definition** (Productoid of an Effectoid \(\langle E, \epsilon \mapsto \cdot, \cdot, \cdot \mapsto \cdot, \cdot \rangle\) for a 2-Category \(C\)). A tuple \(\langle C, m, \mu_\epsilon, \mu_{\leq}, \mu_i, c \rangle\) whose components have the following types:

**Object** \(C\) is an object of \(C\)

**Morphisms** \(m\) maps each \(\epsilon : E\) to a morphism \(m_\epsilon\) of \(C\) from \(C\) to \(C\)

**Unit** \(\mu_\epsilon\) maps each \(\epsilon : E\) satisfying \(\epsilon \mapsto \epsilon\) to a 2-cell \(\mu_\epsilon^\epsilon\) of \(C\) from \(\text{id}_C \Rightarrow m_\epsilon\)

**Coercion** \(\mu_{\leq}\) maps each \(\epsilon, \epsilon' : E\) satisfying \(\epsilon \leq \epsilon'\) to a 2-cell \(\mu_{\leq}^\epsilon_{\epsilon'}\) of \(C\) from \(m_\epsilon \Rightarrow m_{\epsilon'}\)

**Join** \(\mu_i\) maps each \(\epsilon_1, \epsilon_2, \epsilon : E\) satisfying \(\epsilon_1 \delta_\epsilon \epsilon_2 \mapsto \epsilon\) to a 2-cell \(\mu_{\epsilon_1, \epsilon_2}^\epsilon\) of \(C\) from \(m_{\epsilon_1} ; m_{\epsilon_2} \Rightarrow m_\epsilon\)
Coherence $\epsilon$ is a proof that $\mu_{\leq}^{\epsilon}$ always equals $id_{m_{\epsilon}}$ and the following equalities all hold whenever well defined:

First, $C$ and $C$ equal $C$.

In other words, $C$ and $C$ equal $C$.

Second, $C$ equals $C$.

In other words, $C$ equals $C$.

Third, $id_{C} \Rightarrow m_{\epsilon} \Rightarrow m_{\epsilon}'$ equals $id_{C} \Rightarrow m_{\epsilon} \Rightarrow m_{\epsilon}'$.

Fourth, $m_{\epsilon_{1}} : m_{\epsilon_{2}} \Rightarrow m_{\epsilon} \Rightarrow m_{\epsilon'}$ equals $m_{\epsilon_{1}} : m_{\epsilon_{2}} \Rightarrow m_{\epsilon} \Rightarrow m_{\epsilon'}$.

**Theorem.** If an effector is semi-strict, then there is a bijection between the set of productors for that effector and the set of productoids for the effectoid corresponding to that semi-strict effector. The bijection preserves the object $C$ and the mapping $m$. The 2-cells $\mu_{\epsilon}$ correspond to the 2-cells $\mu_{1}^{\epsilon}$; the 2-cells $\mu_{\leq}^{\epsilon}$ correspond to the 2-cells $\mu_{1,[\epsilon]}^{\epsilon}$; and the 2-cells $\mu_{\epsilon_{1},\epsilon_{2}}^{\epsilon}$ correspond to the 2-cells $\mu_{1,\epsilon_{2}}^{\epsilon}$.

**Definition** (Postmodule of a Productor). A tuple $\langle R, r, \rho, d \rangle$ whose components have the following types:

<table>
<thead>
<tr>
<th>Object $R$:</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Morphism $r$:</td>
<td>$C \rightarrow R$</td>
</tr>
<tr>
<td>Action $\rho$:</td>
<td>A mapping from each $\epsilon : E$ to a 2-cell $\rho_{\epsilon} : m_{\epsilon} ; r \Rightarrow r$</td>
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<tr>
<td>Distributivity $d$:</td>
<td>A proof that $C$ equals $R$.</td>
</tr>
</tbody>
</table>