Opfibrations

Ross Tate

November 21, 2014

Definition (Opcartesian Morphism for a functor $F : \mathbf{C} \to \mathbf{D}$)**.** A morphism $c_1 : \mathcal{C} \to \mathcal{C}_1$ of $\mathbf{C}$ with the property that, for every $\mathbf{C}$-morphism $c_2 : \mathcal{C} \to \mathcal{C}_2$ and $\mathbf{D}$-morphism $d : F(\mathcal{C}_1) \to F(\mathcal{C}_2)$ such that $F(c_2)$ equals $F(c_1) \,;\, d$, there exists a unique $c : \mathcal{C}_1 \to \mathcal{C}_2$ such that $F(c)$ equals d and $c_1 \,;\, c$ equals c_2.

Example. A morphism $f : \mathcal{R}_1 \to \mathcal{R}_2$ of $\mathbf{Rel}(2)$ is opcartesian iff

$$\forall r_2, r_2' : R_2.\ r_2 \leq r_2' \implies \exists r_1, r_1' : R_1.\ r_1 \leq r_2 \wedge f(r_1) = r_2 \wedge f(r_1') = r_2'$$

Definition (Opfibration)**.** A functor $F : \mathbf{C} \to \mathbf{D}$ with the property that, for every F-costructured arrow $d : F(\mathcal{C}_1) \to \mathcal{D}_2$, there exists some object $\mathcal{C}_2$ and opcartesian morphism $c : \mathcal{C}_1 \to \mathcal{C}_2$ such that $F(c)$ equals d.

Remark. The term F-costructured arrow was introduced in the Transpositions lecture notes.

Example. The underlying functor for $\mathbf{Rel}(2)$ is an opfibration. Given an object $\langle X, \leq \rangle$ of $\mathbf{Rel}(2)$ and a function $f : X \to Y$, the corresponding opcartesian morphism f is the relation-preserving function from $\langle X, \leq \rangle$ to $\langle Y, \sqsubseteq \rangle$ where $y \sqsubseteq y'$ is defined as $\exists x, x' : X.\ x \leq x' \wedge f(x) = y \wedge f(x') = y'$.

*Remark. Cartesian*ness is dual to opcartesianness. An initial morphism in **Prost** is the same as a cartesian morphism for the underyling functor. The proofs for epi-initial-mono factorizations and unique diagonalizations in **Prost** implicitly relied on cartesianness.