Definition ((Biased) Semigroup). A tuple \( \langle S, \ast, a \rangle \) where the components have the following types:

Underlying Set \( S \): Type
Operator \( \ast: S \times S \to S \) (infix)
Associativity \( a: \forall s_1, s_2, s_3: S. (s_1 \ast s_2) \ast s_3 = s_1 \ast (s_2 \ast s_3) \)

Example. \( N_{\min} = \langle \mathbb{N}, \min, \cdot \rangle \) where \( \mathbb{N}_{\min} \) denotes nonempty (finite) lists of \( T \)
\( (L_{\min})_{++} = \langle L_{\min}, +, +, \cdot \rangle \) where \( L_{\min} \) denotes nonempty multisets of \( T \)
\( (S_{\min})_{\cup} = \langle S_{\min}, \cup, \cdot \rangle \) where \( S_{\min} \) denotes nonempty finite subsets of \( T \)
\( (P_{\min})_{\cap} = \langle P_{\min}, \cap, \cdot \rangle \) where \( P_{\min} \) denotes nonempty subsets of \( T \)

Definition ((Biased) Group Homomorphism from \( \langle S, \ast, a \rangle \) to \( \langle T, +, i, -1, \cdot \rangle \)). A tuple \( \langle f, d, i, \cdot \rangle \) where the components have the following types:

Underlying Function \( f: S \to T \)
Distributivity \( \cdot: \forall s_1, s_2: S. f(s_1) + f(s_2) = f(s_1 \ast s_2) \)

Exercise 1. Prove that \( \text{Mon} \) is a non-full subcategory of \( \text{Sgr} \), the category of semigroups and semigroup homomorphisms (with the obvious composition and identity), via the obvious inclusion functor.

Definition ((Biased) Group). A tuple \( \langle G, *, e, \ast, i, -1, \cdot \rangle \) where the components have the following types:

Underlying Set \( G \): Type
Operator \( \ast: G \times G \to G \) (infix)
Associativity \( a: \forall g_1, g_2, g_3: G. (g_1 \ast g_2) \ast g_3 = g_1 \ast (g_2 \ast g_3) \)
Identity Element \( e: G \)
Identity \( i: \forall g: G. e \ast g = g = g \ast e \)
Inverse Operator \( -1: G \to G \) (postfix)
Inverse \( \cdot\): \( \forall g: G. g \ast g^{-1} = e = g^{-1} \ast g \)

Definition ((Biased) Group Homomorphism from \( \langle G, *, e, \ast, i, -1, \cdot \rangle \) to \( \langle H, +, i, -1, \cdot \rangle \)). A tuple \( \langle f, \cdot, i, \cdot \rangle \) where the components have the following types:
Underlying Function $f : G \to H$

Distributivity $\forall g_1, g_2 : G. f(g_1) + f(g_2) = f(g_1 \ast g_2)$

Identity $i : i = f(e)$

Inverse $\forall g : G. -f(g) = f(g^{-1})$

Exercise 2. Prove that $\textbf{Grp}$, the category of groups and group homomorphism (with the obvious composition and identity), is a full subcategory of $\textbf{Mon}$.

Definition (Reflection Arrow for $S \hookrightarrow I \hookrightarrow C$). An object $C$ of $C$ and object $R$ of $S$ with a morphism $C \xrightarrow{C} I(R)$ of $C$ such that for every object $S$ of $S$ with a morphism $C \xrightarrow{m} I(S)$ of $C$ there exists a unique morphism $R \xrightarrow{m} S$ of $S$ with $r; I(m^\sim) = m$.

Exercise 3. Prove that, for $\textbf{Grp} \hookrightarrow \textbf{Mon}$, the identity on a monoid is a reflection arrow if and only if that monoid is a group.

Exercise 4. Prove that, for $\textbf{Mon} \hookrightarrow \textbf{Sgr}$, the identity on a semigroup is never a reflection arrow even if that semigroup is a monoid.

Definition (Reflective Subcategory). A subcategory $S \hookrightarrow I \hookrightarrow C$ with a reflection arrow $C \xrightarrow{C} I(R)$ for every object $C$ of $C$.

Exercise 5. Prove that $\textbf{Grp}$ is a reflective subcategory of $\textbf{Mon}$, and that $\textbf{Mon}$ is a reflective subcategory of $\textbf{Sgr}$.

Exercise 6. Prove that every reflective subcategory $S \hookrightarrow I \hookrightarrow C$ has a unique way to extend a function $R(C) = R_C$ to a functor so that the following diagram commutes for every morphism $C_1 \xrightarrow{m} C_2$ of $C$ (meaning all paths are equal):

$$
\begin{array}{ccc}
C_1 & \xrightarrow{r_{C_1}} & I(R(C_1)) \\
m & \downarrow & I(R(m)) \\
C_2 & \xrightarrow{r_{C_2}} & I(R(C_2))
\end{array}
$$

Exercise 7. Prove that, for a reflective subcategory $S \hookrightarrow I \hookrightarrow C$, the subcategory $(S, I)$ is full if and only if for every object $S$ of $S$ the identity on $I(S)$ is a reflection arrow.