Nulls

Ross Tate

September 18, 2014

Definition ((Biased) Semigroup)**.** A tuple $\langle S, *, \mathfrak{a}\rangle$ where the components have the following types:

Underlying Set S**:** `Type`

Operator $*$**:** $S \times S \to S$ (infix)

Associativity $\mathfrak{a}$**:** $\forall s_1, s_2, s_3 : S.\ (s_1 * s_2) * s_3 = s_1 * (s_2 * s_3)$

Example. $\mathbb{N}_{\min} = \langle \mathbb{N}, \min, \cdot\rangle$
$(\mathbb{L}_+T)_{++} = \langle \mathbb{L}_+T, +\!\!+, \cdot\rangle$ where $\mathbb{L}_+T$ denotes nonempty (finite) lists of T
$(\mathbb{M}_+T)_+ = \langle \mathbb{M}_+T, +, \cdot\rangle$ where $\mathbb{M}_+T$ denotes nonempty multisets of T
$(\mathbb{S}_+T)_\cup = \langle \mathbb{S}_+T, \cup, \cdot\rangle$ where $\mathbb{S}_+T$ denotes nonempty finite subsets of T
$(\mathbb{S}_+T)_\cap = \langle \mathbb{S}_+T, \cap, \cdot\rangle$
$(\mathbb{P}_+T)_\cup = \langle \mathbb{P}_+T, \cup, \cdot\rangle$ where $\mathbb{P}_+T$ denotes nonempty subsets of T
$(\mathbb{P}_+T)_\cap = \langle \mathbb{P}_+T, \cap, \cdot\rangle$

Definition ((Biased) Semigroup Homomorphism from $\langle S, *, \cdot\rangle$ to $\langle T, +, \cdot\rangle$)**.** A tuple $\langle f, \mathfrak{d}\rangle$ where the components have the following types:

Underlying Function f**:** $S \to T$

Distributivity $\mathfrak{d}$**:** $\forall s_1, s_2 : S.\ f(s_1) + f(s_2) = f(s_1 * s_2)$

Example. $\langle \lambda n.\ n+1, \cdot\rangle$ from $\mathbb{N}_{\min}$ to $\mathbb{N}_{\min}$ (and from $\mathbb{N}_{\max}$ to $\mathbb{N}_{\max}$).

Exercise 1. Prove that **Mon** is a non-full subcategory of **Sgr**, the category of semigroups and semigroup homomorphisms (with the obvious composition and identity), via the obvious inclusion functor.

Definition ((Biased) Group)**.** A tuple $\langle G, *, \mathfrak{a}, e, \mathfrak{i}, ^{-1}, \mathfrak{inv}\rangle$ where the components have the following types:

Underlying Set G**:** `Type`

Operator $*$**:** $G \times G \to G$ (infix)

Associativity $\mathfrak{a}$**:** $\forall g_1, g_2, g_3 : G.\ (g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$

Identity Element e**:** G

Identity $\mathfrak{i}$**:** $\forall g : G.\ e * g = g = g * e$

Inverse Operator $^{-1}$**:** $G \to G$ (postfix)

Inverse $\mathfrak{inv}$**:** $\forall g : G.\ g * g^{-1} = e = g^{-1} * g$

Definition ((Biased) Group Homomorphism from $\langle G, *, \cdot, e, \cdot, ^{-1}, \cdot\rangle$ to $\langle H, +, \cdot, i, \cdot, -, \cdot\rangle$)**.** A tuple $\langle f, \mathfrak{d}, \mathfrak{i}, \mathfrak{inv}\rangle$ where the components have the following types:

Underlying Function f**:** $G \to H$

Distributivity $\mathfrak{d}$**:** $\forall g_1, g_2 : G.\ f(g_1) + f(g_2) = f(g_1 * g_2)$

Identity $\mathfrak{i}$**:** $i = f(e)$

Inverse $\mathfrak{inv}$**:** $\forall g : G.\ -f(g) = f(g^{-1})$

Exercise 2. Prove that **Grp**, the category of groups and group homomorphism (with the obvious composition and identity), is a full subcategory of **Mon**.

Definition (Reflection Arrow for $\mathbf{S} \overset{I}{\hookrightarrow} \mathbf{C}$)**.** An object $\mathcal{C}$ of **C** and object $\mathcal{R}$ of **S** with a morphism $\mathcal{C} \overset{r}{\to} I(\mathcal{R})$ of **C** such that for every object $\mathcal{S}$ of **S** with a morphism $\mathcal{C} \overset{m}{\to} I(\mathcal{S})$ of **C** there exists a unique morphism $\mathcal{R} \overset{m^{\leftarrow}}{\longrightarrow} \mathcal{S}$ of **S** with $r \mathbin{;} I(m^{\leftarrow}) = m$.

Exercise 3. Prove that, for $\mathbf{Grp} \hookrightarrow \mathbf{Mon}$, the identity on a monoid is a reflection arrow if and only if that monoid is a group.

Exercise 4. Prove that, for $\mathbf{Mon} \hookrightarrow \mathbf{Sgr}$, the identity on a semigroup is never a reflection arrow even if that semigroup is a monoid.

Definition (Reflective Subcategory)**.** A subcategory $\mathbf{S} \overset{I}{\hookrightarrow} \mathbf{C}$ with a reflection arrow $\mathcal{C} \overset{r_{\mathcal{C}}}{\to} I(\mathcal{R}_{\mathcal{C}})$ for every object $\mathcal{C}$ of **C**.

Exercise 5. Prove that **Grp** is a reflective subcategory of **Mon**, and that **Mon** is a reflective subcategory of **Sgr**.

Exercise 6. Prove that every reflective subcategory $\mathbf{S} \overset{I}{\hookrightarrow} \mathbf{C}$ has a unique way to extend a function $R(\mathcal{C}) = \mathcal{R}_{\mathcal{C}}$ to a functor so that the following diagram commutes for every morphism $\mathcal{C}_1 \overset{m}{\to} \mathcal{C}_2$ of **C** (meaning all paths are equal):

$$\begin{array}{ccc} \mathcal{C}_1 & \overset{r_{\mathcal{C}_1}}{\longrightarrow} & I(R(\mathcal{C}_1)) \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle I(R(m))} \\ \mathcal{C}_2 & \underset{r_{\mathcal{C}_2}}{\longrightarrow} & I(R(\mathcal{C}_2)) \end{array}$$

Exercise 7. Prove that, for a reflective subcategory $\mathbf{S} \overset{I}{\hookrightarrow} \mathbf{C}$, the subcategory $\langle \mathbf{S}, I \rangle$ is full if and only if for every object $\mathcal{S}$ of **S** the identity on $I(\mathcal{S})$ is a reflection arrow.