# Monads

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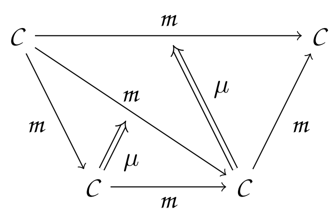
October 16, 2014

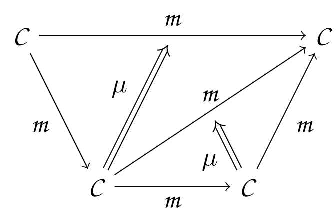
**Definition** (Monad for a 2-Category **C**)**.** A tuple $\langle \mathcal{C}, m, \mu, \mathfrak{a}, \eta, \mathfrak{i}\rangle$ whose components have the following types:

**Object** $\mathcal{C}$ is an object of **C**

**Morphism** $m$ is a morphism of **C** from $\mathcal{C}$ to $\mathcal{C}$

**Join** $\mu$ is a 2-cell of **C** from $m\,;m \Rightarrow m$

**Associativity** $\mathfrak{a}$ is a proof that 

equals 

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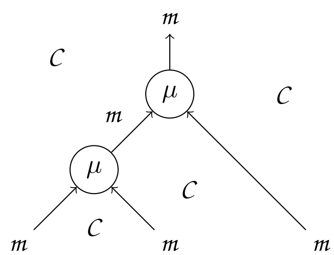
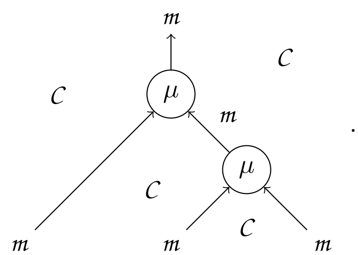
In other words, 

equals 

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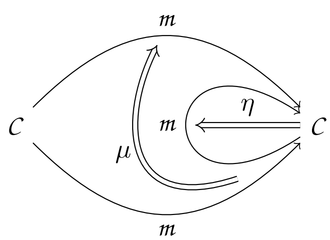
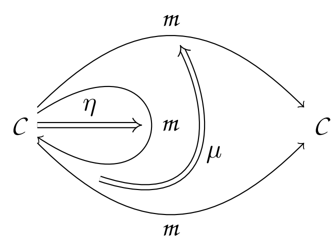
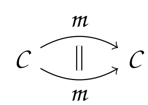
**Unit** $\eta$ is a 2-cell of **C** from $\mathcal{C}$ to $m$

**Identity** $\mathfrak{i}$ is a proof that 

and 

equal 

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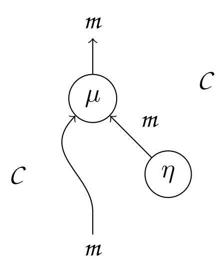
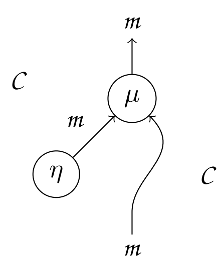
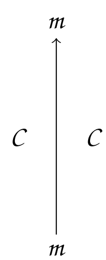
In other words, 

and 

equal 

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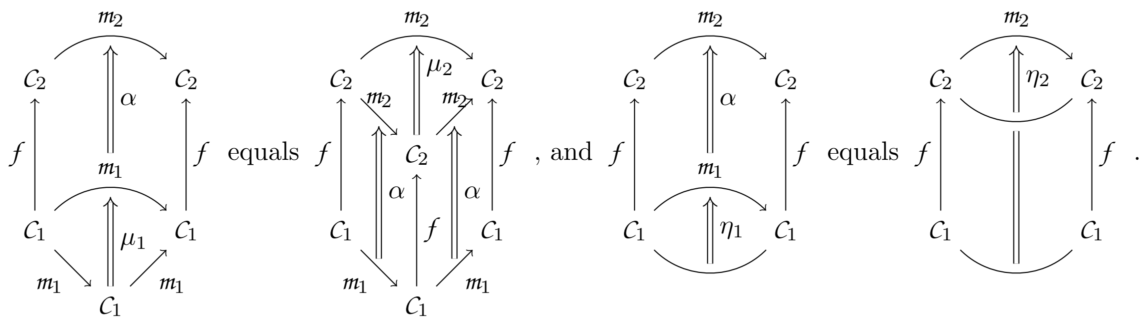
*Remark.* Given a 2-category, one can construct a multicategory whose objects are the 1-cells of the multicategory and whose morphisms are 2-cells from the composition of the inputs to the output. A monad is an internal monoid of that multicategory.

**Theorem.** *For any monad $\langle \mathcal{C}, m, \mu, \text{.}, \eta, \text{.}\rangle$ and $n : \mathbb{N}$, all 2-cells from $m^n$ to $m$ built from $\mu$, $\eta$, and identities are equal.*

**Example.** $\langle \mathbf{Set}, \mathbb{L}, \texttt{flatten}, \blacksquare, [\bullet], \blacksquare \rangle$ is a monad in **CAT**. Similar monads on **Set** exist for $\mathbb{M}$, $\mathbb{S}$, and $\mathbb{P}$.

**Definition** (Monad Morphism from $\langle \mathcal{C}_1, m_1, \mu_1, \blacksquare, \eta_1, \blacksquare \rangle$ to $\langle \mathcal{C}_2, m_2, \mu_2, \blacksquare, \eta_2, \blacksquare \rangle$)**.** A morphism $f : \mathcal{C}_1 \to \mathcal{C}_2$ and a 2-cell $\alpha : m_1 \,; f \Rightarrow f \,; m_2$ such that:




*Remark.* Note that, if $f$ above is required to be an identity morphism, then the above definition corresponds to a morpishm of an internal monoids of a multicategory. The generality above comes from viewing monads as internal monoids of an opetory.

**Example.** The obvious natural transformations from $\mathbb{L}$ to $\mathbb{M}$ to $\mathbb{S}$ to $\mathbb{P}$ are all monad morphisms where $f$ is the identity functor of **Set**.