Monads
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**Definition** (Monad for a 2-Category \( C \)). A tuple \( (C, m, \mu, a, \eta, i) \) whose components have the following types:

- **Object** \( C \) is an object of \( C \)
- **Morphism** \( m \) is a morphism of \( C \) from \( C \) to \( C \)
- **Join** \( \mu \) is a 2-cell of \( C \) from \( m ; m \Rightarrow m \)

**Associativity** \( a \) is a proof that

\[
\begin{align*}
\mu & \quad \mu \\
C & \quad C \\
\mu & \quad \mu \\
C & \quad C \\
m & \quad m \\
m & \quad m \\
m & \quad m \\
C & \quad C
\end{align*}
\]

In other words,

\[
\begin{align*}
\mu & \quad \mu \\
C & \quad C \\
\mu & \quad \mu \\
C & \quad C \\
m & \quad m \\
m & \quad m \\
m & \quad m \\
C & \quad C
\end{align*}
\]

**Unit** \( \eta \) is a 2-cell of \( C \) from \( C \) to \( m \)

**Identity** \( i \) is a proof that

\[
\begin{align*}
\mu & \quad \eta \\
C & \quad C \\
\mu & \quad \eta \\
C & \quad C \\
m & \quad m \\
m & \quad m \\
m & \quad m \\
C & \quad C
\end{align*}
\]

In other words,

\[
\begin{align*}
\mu & \quad \eta \\
C & \quad C \\
\mu & \quad \eta \\
C & \quad C \\
m & \quad m \\
m & \quad m \\
m & \quad m \\
C & \quad C
\end{align*}
\]

**Remark.** Given a 2-category, one can construct a multicategory whose objects are the 1-cells of the multicategory and whose morphisms are 2-cells from the composition of the inputs to the output. A monad is an internal monoid of that multicategory.

**Theorem.** For any monad \( (C, m, \mu, \ast, \eta, \ast) \) and \( n : \mathbb{N} \), all 2-cells from \( m^n \) to \( m \) built from \( \mu, \eta, \) and identities are equal.
Example. $(\text{Set}, L, \text{flatten}, .[\cdot], .)$ is a monad in $\text{CAT}$. Similar monads on $\text{Set}$ exist for $M$, $S$, and $P$.

Definition (Monad Morphism from $(C_1, m_1, \mu_1, \eta_1, \cdot)$ to $(C_2, m_2, \mu_2, \eta_2, \cdot)$). A morphism $f : C_1 \to C_2$ and a 2-cell $\alpha : m_1 ; f \Rightarrow f ; m_2$ such that:

Remark. Note that, if $f$ above is required to be an identity morphism, then the above definition corresponds to a morphism of an internal monoids of a multicategory. The generality above comes from viewing monads as internal monoids of an opetory.

Example. The obvious natural transformations from $L$ to $M$ to $S$ to $P$ are all monad morphisms where $f$ is the identity functor of $\text{Set}$. 
