Limits

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Definition (Product of $C_1$ and $C_2$, where $C_1$ and $C_2$ are objects of $C$). An object, denoted $C_1 \times C_2$ (although more traditionally with $C_1 \times C_2$), along with morphisms $\pi_1 : C_1 \times C_2 \to C_1$ and $\pi_2 : C_1 \times C_2 \to C_2$ with the property that, for any object $C$ and morphisms $f_1 : C \to C_1$ and $f_2 : C \to C_2$, there exists a unique morphism, denoted $\langle f_1, f_2 \rangle$, making the following diagram commute:

\[
\begin{array}{ccc}
C & \xrightarrow{\langle f_1, f_2 \rangle} & C_1 \times C_2 \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
C_1 & & C_2
\end{array}
\]

Example. In $\bf{Set}$, $A \times B$ with the projection functions is the product of $A$ and $B$. In $\bf{Mon}$, $\langle A, B \rangle$ with the projection homomorphisms is the product of $A$ and $B$. In $\bf{Rel}$, $\langle A \times B, \lambda(p_1, p_2), \pi_1(p_1) R \pi_2(p_2) \land \pi_2(p_1) S \pi_2(p_2) \rangle$ with the relation-preserving projection functions is the product of $\langle A, R \rangle$ and $\langle B, S \rangle$. In $\bf{Cat}$, $A \times B$ with the projection functors is the product of $A$ and $B$. In $\bf{Rel}$, the disjoint union of $A$ and $B$ is the product of $A$ and $B$.

Definition (Terminal Object of $C$). An object, denoted $\top$ (although more traditionally with 1), with the property that, for any object $C$, there exists a unique morphism, denoted $\langle \rangle$, from $C$ to $\top$.

Example. In $\bf{Set}$, any singleton set is a terminal object. In $\bf{Mon}$, any singleton monoid is a terminal monoid. In $\bf{Rel}(2)$, $\langle 2, \top \rangle$ is the terminal binary relation. In $\bf{Cat}$, any category with only one object and one morphism is the terminal category. In $\bf{Rel}$, the empty set is the terminal object.

Notation. The unique morphism from an object $C$ to the terminal object is also denoted with $\downarrow e$.

Definition (Equalizer of morphisms $f_1, f_2 : C_1 \to C_2$). An object $E$ along with a morphism $\pi : E \to C_1$ such that $\pi \cdot f_1 = \pi \cdot f_2$ and with the property that, for any other object $C$ and morphism $f : C \to C_1$ such that $f \cdot f_1 = f \cdot f_2$, there exists a unique morphism $(f) : C \to E$ such that $(f) \cdot \pi = f$.

Example. In $\bf{Set}$, the equalizer of functions $f_1, f_2 : X \to Y$ is the set $\{ x : X \mid f_1(x) = f_2(x) \}$ along with the obvious inclusion function to $X$. In $\bf{Mon}$, one restricts to the operators to the above subset, which ends up still forming a well-defined monoid because the functions are monoid homomorphisms. In $\bf{Rel}(2)$, one restricts the relation to the above subset. In $\bf{Cat}$, one builds the equalizer for the components on objects and then for the components on morphisms and then restricts the operators to those subsets, which ends up forming a well-defined category because of distributivity and identity preservation. $\bf{Rel}$ does not have equalizers for some pairs of binary relations.

Definition (Pullback of morphisms $g_1 : C_1 \to C_3$ and $g_2 : C_2 \to C_3$). An object $P$ along with morphisms $\pi_1 : P \to C_1$ and $\pi_2 : P \to C_2$ such that $\pi_1 : f_1 = \pi_2 : f_2$ and with the property that, for any object $C$ and morphisms $g_1 : C \to C_1$ and $g_2 : C \to C_2$ such that $g_1 : f_1 = g_2 : f_1$, there exists a unique morphism, denoted $\langle g_1, g_2 \rangle$, making the following diagram commute:

\[
\begin{array}{ccc}
P & \xrightarrow{\langle f_1, f_2 \rangle} & C_1 \times C_2 \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
C_1 & & C_2
\end{array}
\]
Example. In \textbf{Set}, the pullback of functions $f_1 : X \to Z$ and $f_2 : Y \to Z$ is the set $\{(x, y) : X \times Y \mid f_1(x) = f_2(y)\}$ along with the obvious projection functions.

**Exercise 1.** Note that the construction of pullbacks in \textbf{Set} is built from a product and an equalizer. Prove that if a category has products for all objects and equalizers for all parallel morphism pairs, then it has pullbacks for all morphism pairs with the same codomain.

**Definition** (Limit of a functor $D : S \to C$). An object $L$ of $C$ along with a natural transformation $\pi : L \Rightarrow D$ with the property that, for any object $C$ and natural transformation $\alpha : C \Rightarrow D$, there exists a unique morphism $\langle \alpha \rangle : C \to L$ such that $\langle \alpha \rangle \circ \pi$ equals $\alpha$.

**Definition** (Scheme and Diagram). Given $D : S \to C$, the category $S$ is called the scheme and the functor $D$ is called the diagram in $C$.

Example. Products correspond to limits of diagrams with scheme 2, the category with 2 objects and only identity morphisms. Terminal objects correspond to limits of the diagram with scheme 0, the category with no objects or morphisms. Equalizers correspond to limits of diagrams with the scheme $\bullet_1 \xrightarrow{=} \bullet_2$. Pullbacks correspond to limits of diagrams with the scheme $\bullet_1 \to \bullet_3 \leftarrow \bullet_2$.

**Exercise 2.** Prove that a category has limits for all diagrams with scheme $S$ if and only if the functor $\Delta$ from $C$ to $S \_ C$, mapping each object to its corresponding constant functor and each morphism to its corresponding constant natural transformation, has a right adjoint.

**Remark.** Given a functor $D : S \to C$, a limit is a functor $L : 1 \to C$ and natural transformation $\pi : \langle \rangle_S : L \Rightarrow D$ with the property that, for any functor $C : 1 \to C$ and natural transformation $\alpha : \langle \rangle_S : L \Rightarrow D$, there exists a unique natural transformation $(\alpha) : C \Rightarrow L$ such that the natural transformation specified in the following diagram equals $\alpha$:

**Definition** (Finitely Complete). A category that has a limit for all diagrams with finite schemes, meaning the scheme has a finite set of objects and morphisms.

**Exercise 3.** Prove that a category is finitely complete if and only if it has a terminal objects, products, and equalizers.

**Definition** (Preserves $S$-Limits). A functor $F : C \to D$ with the property that, for any $D, L$, and $\pi$, if $L : 1 \to C$ and $\pi : \langle \rangle ; L \Rightarrow D$ is a limit of $D : S \to C$, then $L ; F$ and the following natural transformation is a limit of $D ; F$:

**Definition** ((Finitely) Continuous). A functor that preserves all limits is called \textit{continuous}. A functor that preserves all finite limits is called \textit{finitely continuous}.

**Exercise 4.** Prove that every right-adjoint functor is continuous.