Limits

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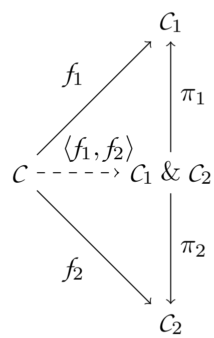
Definition (Product of $\mathcal{C}_1$ and $\mathcal{C}_2$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ are objects of **C**)**.** An object, denoted $\mathcal{C}_1 \,\&\, \mathcal{C}_2$ (although more traditionally with $\mathcal{C}_1 \times \mathcal{C}_2$), along with morphisms $\pi_1 : \mathcal{C}_1 \,\&\, \mathcal{C}_2 \to \mathcal{C}_1$ and $\pi_2 : \mathcal{C}_1 \,\&\, \mathcal{C}_2 \to \mathcal{C}_2$ with the property that, for any object $\mathcal{C}$ and morphisms $f_1 : \mathcal{C} \to \mathcal{C}_1$ and $f_2 : \mathcal{C} \to \mathcal{C}_2$, there exists a unique morphism, denoted $\langle f_1, f_2 \rangle$, making the following diagram commute:



Example. In **Set**, $A \times B$ with the projection functions is the product of A and B. In **Mon**, $\mathcal{A} \& \mathcal{B}$ with the projection homomorphisms is the product of $\mathcal{A}$ and $\mathcal{B}$. In **Rel**(2), $\langle A \times B, \lambda \langle p_1, p_2 \rangle.\ \pi_1(p_1)\ R\ \pi_1(p_2) \wedge \pi_2(p_1)\ S\ \pi_2(p_2) \rangle$ with the relation-preserving projection functions is the product of $\langle A, R \rangle$ and $\langle B, S \rangle$. In **Cat**, $\mathbf{A} \times \mathbf{B}$ with the projection functors is the product of **A** and **B**. In **Rel**, the disjoint union of A and B is the product of A and B.

Definition (Terminal Object of **C**)**.** An object, denoted $\top$ (although more traditionally with 1), with the property that, for any object $\mathcal{C}$, there exists a unique morphism, denoted $\langle\rangle$, from $\mathcal{C}$ to $\top$.

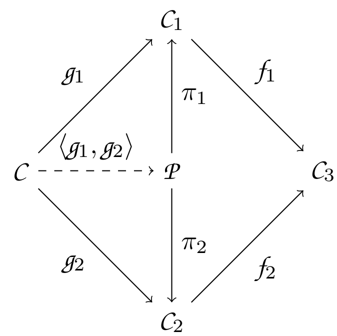
Example. In **Set**, any singleton set is a terminal object. In **Mon**, any singleton monoid is a terminal monoid. In **Rel**(2), $\langle \mathbb{1}, \top \rangle$ is the terminal binary relation. In **Cat**, any category with only one object and one morphism is the terminal category. In **Rel**, the empty set is the terminal object.

Notation. The unique morphism from an object $\mathcal{C}$ to the terminal object is also denoted with $!_{\mathcal{C}}$.

Definition (Equalizer of morphisms $f_1, f_2 : \mathcal{C}_1 \to \mathcal{C}_2$)**.** An object $\mathcal{E}$ along with a morphism $\pi : \mathcal{E} \to \mathcal{C}_1$ such that $\pi \,;f_1 = \pi \,; f_2$ and with the property that, for any other object $\mathcal{C}$ and morphism $f : \mathcal{C} \to \mathcal{C}_1$ such that $f \,; f_1 = f \,; f_2$, there exists a unique morphism $\langle f \rangle : \mathcal{C} \to \mathcal{E}$ such that $\langle f \rangle \,; \pi = f$.

Example. In **Set**, the equalizer of functions $f_1, f_2 : X \to Y$ is the set $\{x : X \mid f_1(x) = f_2(x)\}$ along with the obvious inclusion function to X. In **Mon**, one restricts to the operators to the above subset, which ends up still forming a well-defined monoid because the functions are monoid homomorphisms. In **Rel**(2), one restricts the relation to the above subset. In **Cat**, one builds the equalizer for the components on objects and then for the components on morphisms and then restricts the operators to those subsets, which ends up forming a well-defined category because of distributivity and identity preservation. **Rel** does not have equalizers for some pairs of binary relations.

Definition (Pullback of morhisms $f_1 : \mathcal{C}_1 \to \mathcal{C}_3$ and $f_2 : \mathcal{C}_2 \to \mathcal{C}_3$)**.** An object $\mathcal{P}$ along with morphisms $\pi_1 : \mathcal{P} \to \mathcal{C}_1$ and $\pi_2 : \mathcal{P} \to \mathcal{C}_2$ such that $\pi_1 \,; f_1 = \pi_2 \,; f_2$ and with the property that, for any object $\mathcal{C}$ and morphisms $g_1 : \mathcal{C} \to \mathcal{C}_1$ and $g_2 : \mathcal{C} \to \mathcal{C}_2$ such that $g_1 \,; f_1 = g_2 \,; f_1$, there exists a unique morphism, denoted $\langle g_1, g_2 \rangle$, making the following diagram commute:



Example. In **Set**, the pullback of functions $f_1 : X \to Z$ and $f_2 : Y \to Z$ is the set $\{\langle x, y\rangle : X \times Y \mid f_1(x) = f_2(y)\}$ along with the obvious projection functions.

Exercise 1. Note that the construction of pullbacks in **Set** is built from a product and an equalizer. Prove that if a category has products for all objects and equalizers for all parallel morphism pairs, then it has pullbacks for all morphism pairs with the same codomain.

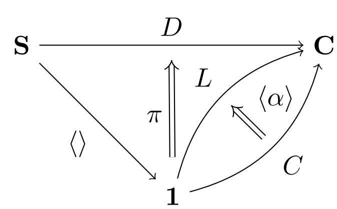
Definition (Limit of a functor $D : \mathbf{S} \to \mathbf{C}$)**.** An object $\mathcal{L}$ of $\mathbf{C}$ along with a natural transformation $\pi : \mathcal{L} \Rightarrow D$ with the property that, for any object $\mathcal{C}$ and natural transformation $\alpha : \mathcal{C} \Rightarrow D$, there exists a unique morphism $\langle\alpha\rangle : \mathcal{C} \to \mathcal{L}$ such that $\langle\alpha\rangle \,;\pi$ equals α.

Definition (Scheme and Diagram)**.** Given $D : \mathbf{S} \to \mathbf{C}$, the category $\mathbf{S}$ is called the scheme and the functor D is called the diagram in $\mathbf{C}$.

Example. Products correspond to limits of diagrams with scheme $\mathbb{2}$, the category with 2 objects and only identity morphisms. Terminal objects correspond to limits of the diagram with scheme $\mathbf{0}$, the category with no objects or morphisms. Equalizers correspond to limits of diagrams with the scheme $\bullet_1 \rightrightarrows \bullet_2$. Pullbacks correspond to limits of diagrams with the scheme $\bullet_1 \to \bullet_3 \leftarrow \bullet_2$.

Exercise 2. Prove that a category has limits for all diagrams with scheme $\mathbf{S}$ if and only if the functor Δ from $\mathbf{C}$ to $\mathbf{S} \multimap \mathbf{C}$, mapping each object to its corresponding constant functor and each morphism to its corresponding constant natural transformation, has a right adjoint.

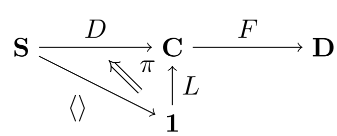
Remark. Given a functor $D : \mathbf{S} \to \mathbf{C}$, a limit is a functor $L : \mathbf{1} \to \mathbf{C}$ and natural transformation $\pi : \langle\rangle_{\mathbf{S}} \,; L \Rightarrow D$ with the property that, for any functor $C : \mathbf{1} \to \mathbf{C}$ and natural transformation $\alpha : \langle\rangle_{\mathbf{S}} \,; L \Rightarrow D$, there exists a unique natural transformation $\langle\alpha\rangle : C \Rightarrow L$ such that the natural transformation specified in the following diagram equals α:



Definition (Finitely Complete)**.** A category that has a limit for all diagrams with finite schemes, meaning the scheme has a finite set of objects and morphisms.

Exercise 3. Prove that a category is finitely complete if and only if it has a terminal objects, products, and equalizers.

Definition (Preserves $\mathbf{S}$-Limits)**.** A functor $F : \mathbf{C} \to \mathbf{D}$ with the property that, for any D, L, and π, if $L : \mathbf{1} \to \mathbf{C}$ and $\pi : \langle\rangle \,; L \Rightarrow D$ is a limit of $D : \mathbf{S} \to \mathbf{C}$, then $L \,; F$ and the following natural transforation is a limit of $D \,; F$:



Definition ((Finitely) Continuous)**.** A functor that preserves all limits is called *continuous*. A functor that preserves all finite limits is called *finitely* continuous.

Exercise 4. Prove that every right-adjoint functor is continuous.