**Definition** ((\(\mathcal{C}, m, \mu, \eta, \iota\))-Postmodule in a 2-Category \(\mathcal{C}\)). A tuple \((\mathcal{R}, r, \rho, \varnothing, \iota)\) whose components have the following types:

- **Object** \(\mathcal{R} : \mathcal{C}\)
- **Morphism** \(r : \mathcal{C} \to \mathcal{R}\)
- **Action** \(\rho : m; r \Rightarrow r\)

**Distributivity** \(\varnothing\): A proof that

\[
\begin{align*}
\mathcal{C} & \xrightarrow{\rho} \mathcal{R} \\
\mathcal{C} & \xrightarrow{m} \mathcal{C} \\
\mathcal{C} & \xrightarrow{r} \mathcal{C}
\end{align*}
\]

equals

\[
\begin{align*}
\mathcal{C} & \xrightarrow{\rho} \mathcal{R} \\
\mathcal{C} & \xrightarrow{m} \mathcal{C} \\
\mathcal{C} & \xrightarrow{r} \mathcal{R}
\end{align*}
\]

In other words,

\[
\begin{align*}
\mathcal{C} & \xrightarrow{\rho} \mathcal{C} \\
\mathcal{C} & \xrightarrow{m} \mathcal{R} \\
\mathcal{C} & \xrightarrow{r} \mathcal{R}
\end{align*}
\]

equals

\[
\begin{align*}
\mathcal{C} & \xrightarrow{\rho} \mathcal{C} \\
\mathcal{C} & \xrightarrow{m} \mathcal{C} \\
\mathcal{C} & \xrightarrow{r} \mathcal{R}
\end{align*}
\]

**Identity** \(\iota\): A proof that

\[
\begin{align*}
\mathcal{C} & \xrightarrow{\eta} \mathcal{R} \\
\mathcal{C} & \xrightarrow{m} \mathcal{C} \\
\mathcal{C} & \xrightarrow{r} \mathcal{C}
\end{align*}
\]

equals

\[
\begin{align*}
\mathcal{C} & \xrightarrow{\rho} \mathcal{R} \\
\mathcal{C} & \xrightarrow{m} \mathcal{C} \\
\mathcal{C} & \xrightarrow{r} \mathcal{R}
\end{align*}
\]

In other words,

\[
\begin{align*}
\mathcal{C} & \xrightarrow{\rho} \mathcal{R} \\
\mathcal{C} & \xrightarrow{m} \mathcal{R} \\
\mathcal{C} & \xrightarrow{r} \mathcal{R}
\end{align*}
\]

Remark. A postmodule is more commonly called a right module.

**Theorem.** For every monad \((\mathcal{C}, m, \mu, \eta, \iota)\), the tuple \((\mathcal{C}, m, \mu, \varnothing, \iota)\) is a postmodule of that monad.

**Definition** (\(\text{Eff}(\mathcal{M})\) where \(\mathcal{M} = (\mathcal{C}, M, \mu, \eta, \iota)\) is a CAT-Monad). A category whose objects are the object of \(\mathcal{C}\) and whose morphisms from \(C_1\) to \(C_2\) are the \(\mathcal{C}\)-morphisms from \(C_1\) to \(M(C_2)\). Given \(f : C_1 \to C_2\) and \(g : C_2 \to C_3\) in \(\text{Eff}(\mathcal{M})\), their composition in \(\text{Eff}(\mathcal{M})\) is the \(\mathcal{C}\)-morphism \(f \circ M(g) \circ \mu_{C_3}\). This composition is associative due to...
naturality and associativity of \( \mu \). Given an object \( C \), the identity morphism in \( \text{Eff}(M) \) is the \( C \)-morphism \( \eta_C \). This is an identity with respect to composition due to identity of \( \eta \) with respect to \( \mu \).

Remark. \( \text{Eff}(M) \) is known as the Kleisli category of \( M \).

Exercise 1. Prove that \( \text{Eff}(P) \) is isomorphic to \( \text{Rel} \).

Exercise 2. Prove that there is a functor \( I : C \rightarrow \text{Eff}(M) \) that maps \( C \) to \( C \) and \( f \) to the morphism whose corresponding \( C \)-morphism is \( f : \eta \) (or equivalently \( \eta ; M(f) \)). Prove that there is a natural transformation \( \varrho : M ; I \Rightarrow I \) that maps \( C \) to the morphism whose corresponding \( C \)-morphism is \( \text{id}_M(C) \). Prove that \( \langle \text{Eff}(M), I, \varrho, \star, \star \rangle \) is a \( M \)-postmodule.

Remark. \( I \) above is injective if and only if \( \eta \) is a natural monomorphism, meaning \( \eta_C \) is a monomorphism for all \( C \).

Exercise 3. Prove that for any \( \text{CAT} \)-monad \( M \) and \( M \)-postmodule \( \langle R, R, \rho, \star, \star \rangle \), there is a unique functor \( R' : \text{Eff}(M) \rightarrow R \) such that \( R = I ; R' \) and \( \rho = \varrho \cdot R' \).

Remark. Given a 2-category \( C \), one can construct an opetory with the same 0-cells and 1-cells and with a 2-cell for each 2-cell from the composition of the inputs to the output. \( 1 \) is the opetory with one 0-cell \( C \), one 1-cell \( m : C \rightarrow C \), and one 2-cell from \( m^n \Rightarrow m \) for each \( n : N \). A monad \( M \) in \( C \) corresponds to a functor \( M \) of opetories from \( 1 \) to \( C \). Let \( 1_r \) be the operator with two 0-cells \( C \) and \( R \), two 1-cells \( m : C \rightarrow C \) and \( r : C \rightarrow R \), and one 2-cell from \( m^n r \) to \( r \) for each \( n : N \) and one 2-cell from \( m^n \Rightarrow m \) for each \( n : N \). There is a unique functor of opetories from \( 1 \) to \( 1_r \), which we will call \( I_r \). An \( M \)-postmodule \( R \), then, corresponds to a functor \( R \) of opetories from \( 1_r \) to \( C \) such that \( I_r ; R \) equals \( M \).

Exercise 4. Show that a monad morphism from \( M_1 \) to \( M_2 \) provides a functor from \( \text{Eff}(M_1) \) to \( \text{Eff}(M_2) \).