# Kleisli Categories

Ross Tate

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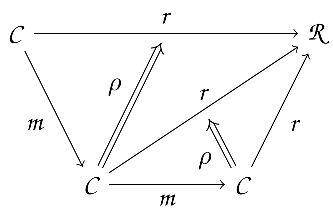
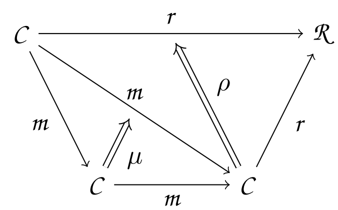
**Definition** ($\langle \mathcal{C}, m, \mu, \centerdot, \eta, \centerdot \rangle$-Postmodule in a 2-Category $\mathbf{C}$)**.** A tuple $\langle \mathcal{R}, r, \rho, \mathfrak{d}, \mathfrak{i} \rangle$ whose components have the following types:

**Object** $\mathcal{R}$**:** $\mathbf{C}$

**Morphism** $r$**:** $\mathcal{C} \to \mathcal{R}$

**Action** $\rho$**:** $m\,;r \Rightarrow r$

**Distributivity** $\mathfrak{d}$**:** A proof that 

equals 

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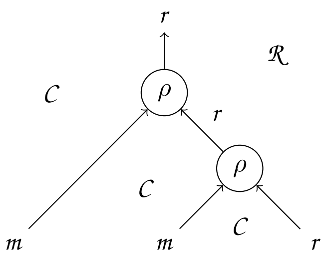
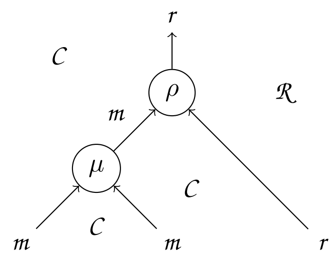
In other words, 

equals 

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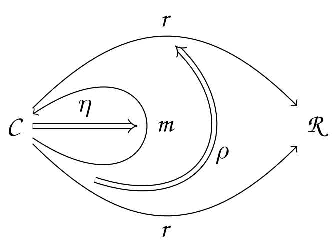
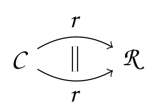
**Identity** $\mathfrak{i}$**:** A proof that 

equals 

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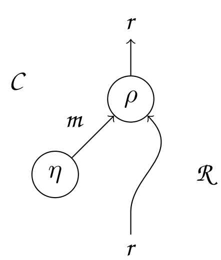
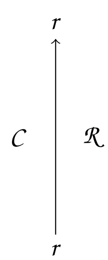
In other words, 

equals 

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*Remark.* A postmodule is more commonly called a right module.

**Theorem.** *For every monad $\langle \mathcal{C}, m, \mu, \mathfrak{d}, \eta, \mathfrak{i} \rangle$, the tuple $\langle \mathcal{C}, m, \mu, \mathfrak{d}, \mathfrak{i} \rangle$ is a postmodule of that monad.*

**Definition** ($\mathbf{Eff}(\mathcal{M})$ where $\mathcal{M} = \langle \mathbf{C}, M, \mu, \centerdot, \eta, \centerdot \rangle$ is a **CAT**-Monad)**.** A category whose objects are the object of $\mathbf{C}$ and whose morphisms from $\mathcal{C}_1$ to $\mathcal{C}_2$ are the $\mathbf{C}$-morphisms from $\mathcal{C}_1$ to $M(\mathcal{C}_2)$. Given $f : \mathcal{C}_1 \to \mathcal{C}_2$ and $g : \mathcal{C}_2 \to \mathcal{C}_3$ in $\mathbf{Eff}(\mathcal{M})$, their composition in $\mathbf{Eff}(\mathcal{M})$ is the $\mathbf{C}$-morphism $f\,; M(g)\,; \mu_{\mathcal{C}_3}$. This composition is associative due to naturality and associativity of $\mu$. Given an object $\mathcal{C}$, the identity morphism in $\mathbf{Eff}(\mathcal{M})$ is the $\mathbf{C}$-morphism $\eta_{\mathcal{C}}$. This is an identity with respect to composition due to identity of $\eta$ with respect to $\mu$.

*Remark.* $\mathbf{Eff}(\mathcal{M})$ is known as the Kleisli category of $\mathcal{M}$.

**Exercise 1.** Prove that $\mathbf{Eff}(\mathbb{P})$ is isomorphic to $\mathbf{Rel}$.

**Exercise 2.** Prove that there is a functor $I : \mathbf{C} \to \mathbf{Eff}(\mathcal{M})$ that maps $\mathcal{C}$ to $\mathcal{C}$ and $f$ to the morphism whose corresponding $\mathbf{C}$-morphism is $f\,;\eta$ (or equivalently $\eta\,;M(f)$). Prove that there is a natural transformation $\varrho : M\,;I \Rightarrow I$ that maps $\mathcal{C}$ to the morphism whose corresponding $\mathbf{C}$-morphism is $id_{M}(\mathcal{C})$. Prove that $\langle \mathbf{Eff}(\mathcal{M}), I, \varrho, \mathbf{.}, \mathbf{.} \rangle$ is a $\mathcal{M}$-postmodule.

*Remark.* $I$ above is injective if and only if $\eta$ is a natural monomorphism, meaning $\eta_{\mathcal{C}}$ is a monomorphism for all $\mathcal{C}$.

**Exercise 3.** Prove that for any $\mathbf{CAT}$-monad $\mathcal{M}$ and $\mathcal{M}$-postmodule $\langle \mathbf{R}, R, \rho, \mathbf{.}, \mathbf{.} \rangle$, there is a unique functor $R' : \mathbf{Eff}(\mathcal{M}) \to \mathbf{R}$ such that $R = I\,;R'$ and $\rho = \varrho \cdot R'$.

*Remark.* Given a 2-category $\mathbf{C}$, one can construct an opetory with the same 0-cells and 1-cells and with a 2-cell for each 2-cell from the composition of the inputs to the output. $\mathbf{1}$ is the opetory with one 0-cell $\mathcal{C}$, one 1-cell $m : \mathcal{C} \to \mathcal{C}$, and one 2-cell from $m^n \Rightarrow m$ for each $n : \mathbb{N}$. A monad $\mathcal{M}$ in $\mathbf{C}$ corresponds to a functor $M$ of opetories from $\mathbf{1}$ to $\mathbf{C}$. Let $\mathbf{1}_r$ be the operatory with two 0-cells $\mathcal{C}$ and $\mathcal{R}$, two 1-cells $m : \mathcal{C} \to \mathcal{C}$ and $r : \mathcal{C} \to \mathcal{R}$, and one 2-cell from $m^n r$ to $r$ for each $n : \mathbb{N}$ and one 2-cell from $m^n \Rightarrow m$ for each $n : \mathbb{N}$. There is a unique functor of opetories from $\mathbf{1}$ to $\mathbf{1}_r$, which we will call $I_r$. An $\mathcal{M}$-postmodule $\mathcal{R}$, then, corresponds to a functor $R$ of opetories from $\mathbf{1}_r$ to $\mathbf{C}$ such that $I_r\,;R$ equals $M$.

**Exercise 4.** Show that a monad morphism from $\mathcal{M}_1$ to $\mathcal{M}_2$ provides a functor from $\mathbf{Eff}(\mathcal{M}_1)$ to $\mathbf{Eff}(\mathcal{M}_2)$.