Homomorphisms

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Definition ((Biased) Monoid Homomorphism from $\langle M, *, \blacksquare, e, \blacksquare\rangle$ to $\langle N, +, \blacksquare, i, \blacksquare\rangle$)**.** A tuple $\langle f, \mathfrak{d}, \mathfrak{i}\rangle$ where the components have the following types:

> **Underlying Function** f**:** $M \to N$
>
> **Distributivity** $\mathfrak{d}$**:** $\forall m_1, m_2 : M.\ f(m_1) + f(m_2) = f(m_1 * m_2)$
>
> **Identity** $\mathfrak{i}$**:** $i = f(e)$

Example. $\langle \lambda\vec{t}.\ \vec{t} \text{ as } \mathbb{M}T, \blacksquare, \blacksquare\rangle$ from $(\mathbb{L}T)_{++}$ to $(\mathbb{M}T)_{+}$

$\langle \det, \blacksquare, \blacksquare\rangle$ from $\mathbb{R}_{\cdot}^{n\times n}$ to $\mathbb{R}_*$ where $\mathbb{R}_{\cdot}^{n\times n}$ is matrices with matrix multiplication and det is the determinant

$\langle \lambda m.\ \lambda\vec{x}.\ M \cdot \vec{x}, \blacksquare, \blacksquare\rangle$ from $\mathbb{R}_{\cdot}^{n\times n}$ to $(\mathbb{R}^n \to \mathbb{R}^n)_{\circ}$

Notation. We will say a function $f : M \to N$ is a biased monoid homomorphism from $\langle M, *, \blacksquare, e, \blacksquare\rangle$ to $\langle N, +, \blacksquare, i, \blacksquare\rangle$ if there exist proofs $\mathfrak{d}$ and $\mathfrak{i}$ such that $\langle f, \mathfrak{d}, \mathfrak{i}\rangle$ is a biased monoid homomorphism from $\langle M, *, \blacksquare, e, \blacksquare\rangle$ to $\langle N, +, \blacksquare, i, \blacksquare\rangle$.

Definition ((Unbiased) Monoid Homomorphism from $\langle M, \Pi, \blacksquare, \blacksquare\rangle$ to $\langle N, \Sigma, \blacksquare, \blacksquare\rangle$)**.** A tuple $\langle f, \mathfrak{d}\rangle$ where the components have the following types:

> **Underlying Function** f**:** $M \to N$
>
> **Distributivity** $\mathfrak{d}$**:** $\forall n : \mathbb{N},\ m_1, \ldots, m_n : M.\ \Sigma\,[f(m_1), \ldots, f(m_n)] = f\,(\Pi\,[m_1, \ldots, m_n])$

Example. $\langle -, \blacksquare\rangle$ from $\mathbb{Z}_\Sigma$ to $\mathbb{Z}_\Sigma$

$\langle -, \blacksquare\rangle$ from $\mathbb{R}_\Sigma$ to $\mathbb{R}_\Sigma$

$\langle -, \blacksquare\rangle$ from $\mathbb{N}_\Sigma$ to $\mathbb{Z}_\Sigma$

$\langle -, \blacksquare\rangle$ from $\mathbb{Z}_{\min}^{+\infty}$ to $\mathbb{Z}_{\max}^{-\infty}$

$\langle -, \blacksquare\rangle$ from $\mathbb{Z}_{\max}^{-\infty}$ to $\mathbb{Z}_{\min}^{+\infty}$

$\langle |\ |, \blacksquare\rangle$ from $\mathbb{Z}_\Pi$ to $\mathbb{N}_\Pi$

$\langle \lambda n.\ n \text{ as } \mathbb{Z}, \blacksquare\rangle$ from $\mathbb{N}_\Sigma$ to $\mathbb{Z}_\Sigma$

$\langle \lambda i.\ i \text{ as } \mathbb{R}, \blacksquare\rangle$ from $\mathbb{Z}_\Pi$ to $\mathbb{R}_\Pi$

Notation. We will say a function $f : M \to N$ is an unbiased monoid homomorphism from $\langle M, \Pi, \blacksquare, \blacksquare\rangle$ to $\langle N, \Sigma, \blacksquare, \blacksquare\rangle$ if there exists some proof $\mathfrak{d}$ such that $\langle f, \mathfrak{d}\rangle$ is an unbiased monoid homomorphism from $\langle M, \Pi, \blacksquare, \blacksquare\rangle$ to $\langle N, \Sigma, \blacksquare, \blacksquare\rangle$.

Exercise 1. Suppose that $\mathcal{M}_U$ and $\mathcal{M}_B$ are an unbiased monoid and a biased monoid with the same underlying set M and with $\mathit{Bias}(\mathcal{M}_U) = \mathcal{M}_B$ and $\mathit{Unbias}(\mathcal{M}_B) = \mathcal{M}_U$, and suppose that $\mathcal{N}_U$ and $\mathcal{N}_B$ are an unbiased monoid and a biased monoid with the same underlying set N and with $\mathit{Bias}(\mathcal{N}_U) = \mathcal{N}_B$ and $\mathit{Unbias}(\mathcal{N}_B) = \mathcal{N}_U$. Prove for any function $f : M \to N$, that f is an unbiased monoid homomorphism from $\mathcal{M}_U$ to $\mathcal{N}_U$ if and only if f is a biased monoid homomorphism from $\mathcal{M}_B$ to $\mathcal{N}_B$.