Enrichment

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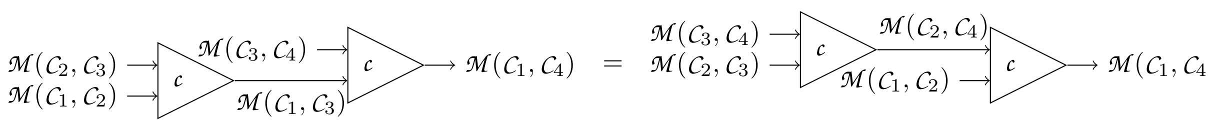
Definition ((Biased) **M**-Enriched Category where **M** is a Multicategory)**.** A tuple $\langle O, \mathcal{M}, c, \mathfrak{a}, i, \mathfrak{i} \rangle$ where the components have the following types:

Objects O**:** $\texttt{Type}_1$

Morphisms $\mathcal{M}$**:** For each pair of objects $C_1, C_2 : O$, an object $\mathcal{M}(C_1, C_2)$ of **M**

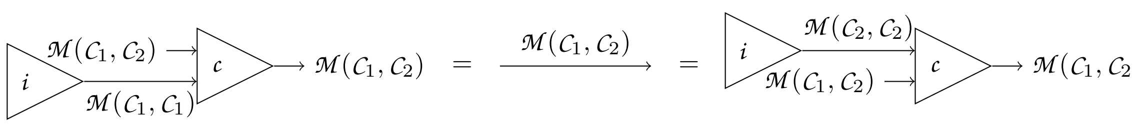
Compositions c**:** For each triple of objects $C_1, C_2, C_3 : O$, an **M**-morphism $c : [\mathcal{M}(C_1, C_2), \mathcal{M}(C_2, C_3)] \to \mathcal{M}(C_1, C_3)$

Associativity $\mathfrak{a}$**:** For each quadruple of objects $C_1, C_2, C_3, C_4 : O$,



Identities i**:** For each object $C : O$, an **M**-morphism $i : [\,] \to \mathcal{M}(C, C)$

Identity $\mathfrak{i}$**:** For each pair of object $C_1, C_2 : O$,



Example. The following are equivalent to categories enriched in a respective multicategory:

Set: A category

Prost: A category plus a preorder $\leq$ on each set of morphisms such that $f_1 \leq f_2 \wedge g_1 \leq g_2 \implies f_1 \,;g_1 \leq f_2 \,;g_2$

$\langle \mathbb{R}_{\geq 0}, +, 0, \geq \rangle$**:** A hemimetric space: a set of "locations" L and a function d specifying the "distance" from one location to another such that $0 \geq d(\ell, \ell)$ and $d(\ell_1, \ell_2) + d(\ell_2, \ell_3) \geq d(\ell_1, \ell_3)$

Definition ((Unbiased) **M**-Enriched Category)**.** A pair $\langle O, C \rangle$ where $O : \texttt{Type}_1$ and C is a functor of multicategories from $\mathbf{Path}(O)$ to **M**.

Exercise 1. Give a bijection between biased and unbiased **M**-enriched categories.

Remark. The multicategory **Prost** has an "underlying" functor U to the multicategory **Set**. We say a category **C** (enriched in **Set**) *can be enriched* in **Prost** when its functor C from $\mathbf{Path}(O_{\mathbf{C}})$ to **Set** can *factor through* U, meaning there is a functor C' from $\mathbf{Path}(O_{\mathbf{C}})$ to **Prost** such that $C' \,; U = C$.

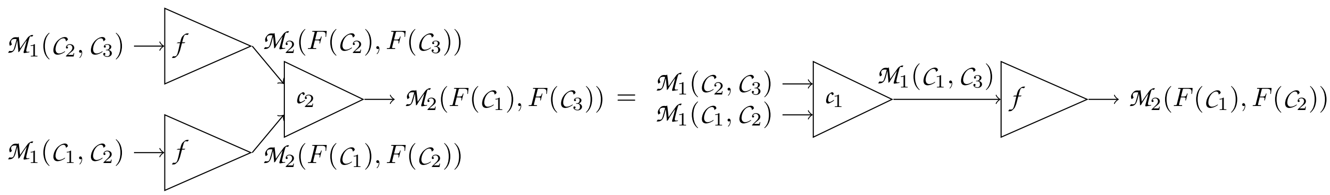
Example. The category **Prost** can be enriched in the multicategory **Prost**. For a pair of objects $\langle X, \leq \rangle$ and $\langle Y, \leq \rangle$ of the category **Prost**, define $\mathcal{M}(\langle X, \leq \rangle, \langle Y, \leq \rangle)$ to be $M_{\mathbf{Prost}}(\langle X, \leq \rangle, \langle Y, \subseteq \rangle)$ equipped with the preorder $f \leq g$ defined by $\forall x.\ f(x) \leq g(x)$. Next we need to show that composition preserves this preorder on morphisms. If $f_1 \leq f_2$ and $g_1 \leq g_2$ then $\forall x.\ (f_1 \,; g_1)(x) = g_1(f_1(x)) \leq g_1(f_2(x)) \leq g_2(f_2(x)) = (f_2 \,; g_2)(x)$, so $f_1 \,; g_1 \leq g_2 \,; g_2$. Identity is trivially relation-preserving.

Definition ((Biased) **M**-Enriched Functor from $\langle O_1, \mathcal{M}_1, c_1, \centerdot, i_1, \centerdot\rangle$ to $\langle O_2, \mathcal{M}_2, c_2, \centerdot, i_2, \centerdot\rangle$)**.** A tuple $\langle F, f, \mathfrak{d}, \mathfrak{i}\rangle$ where the components have the following types:

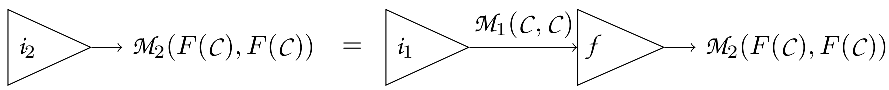
Object Mapping F**:** $O_1 \to O_2$

Morphism "Mapping" f**:** For all pairs $\mathcal{C}_1, \mathcal{C}_2 : O_1$, an **M**-morphism $f : [\mathcal{M}_1(\mathcal{C}_1, \mathcal{C}_2)] \to \mathcal{M}_2(F(\mathcal{C}_1), F(\mathcal{C}_2))$

Distributivity $\mathfrak{d}$**:** For all triples of objects $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 : O_1$:



Identity $\mathfrak{i}$**:** For all object $\mathcal{C} : O$,



Example. A $\langle \mathbb{R}_{\geq 0}, +, 0, \geq\rangle$-enriched functor is a nonexpansive map from a hemimetric space with location set L_1 and distance function d_1 to a hemimetric space with location set L_2 and distance function d_2, meaning a function $f : L_1 \to L_2$ such that for all locations $\ell_1, \ell_2 : L_1$ we have $d_1(\ell_1, \ell_2) \geq d_2(f(\ell_1), f(\ell_2))$.