Definition (Effector). A tuple \((E, \to, a, i)\) whose components have the following types:

<table>
<thead>
<tr>
<th>Set of Effects (E): Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sequence Relation (\to): (L(E) \times E \to \text{Prop})</td>
</tr>
<tr>
<td>Associativity (a): (\forall \varepsilon_1, \ldots, \varepsilon_n, \varepsilon. \forall \varepsilon_1, \ldots, \varepsilon_n. (\forall i. \varepsilon_i \to \varepsilon_i) \land [\varepsilon_1, \ldots, \varepsilon_n] \to \varepsilon \implies \varepsilon_1 + + + \varepsilon_n \to \varepsilon)</td>
</tr>
<tr>
<td>Identity (i): (\forall \varepsilon. [\varepsilon] \to \varepsilon)</td>
</tr>
</tbody>
</table>

Example. An important example is where \(E\) is the singleton set and \(\to\) always holds.

Definition ((Biased) M-Enriched Natural Transformation from \(\langle F_1, f_1, \ldots \rangle\) to \(\langle F_2, f_2, \ldots \rangle\) as \(M\)-enriched functors from \(\langle O_1, M_1, C_1, \ldots, i_1, \ldots \rangle\) to \(\langle O_2, M_2, C_2, \ldots, i_2, \ldots \rangle\). A tuple \(\langle t, n \rangle\) where the components have the following types:

| Transformation \(t\): For all pairs \(C_1, C_2 : O_1\), an \(M\)-morphism \(t : [M_1(C_1, C_2)] \to M_2(F_1(C_1), F_2(C_2))\) |
| Naturality \(\delta\): For all triples of objects \(C_1, C_2, C_3 : O_1\): |

\[
\begin{align*}
\mathcal{M}_1(C_2, C_3) & \xrightarrow{f_2} \mathcal{M}_2(F_2(C_2), F_2(C_3)) \\
\mathcal{M}_1(C_1, C_2) & \xrightarrow{t} \mathcal{M}_2(F_1(C_1), F_2(C_2)) \\
\mathcal{M}_1(C_2, C_3) & \xrightarrow{c_2} \mathcal{M}_2(F_1(C_1), F_2(C_3)) \\
\mathcal{M}_1(C_1, C_2) & \xrightarrow{f_1} \mathcal{M}_2(F_1(C_1), F_1(C_2)) \\
\mathcal{M}_1(C_2, C_3) & \xrightarrow{e_2} \mathcal{M}_2(F_1(C_1), F_2(C_3)) \\
\mathcal{M}_1(C_1, C_2) & \xrightarrow{e_1} \mathcal{M}_2(F_1(C_1), F_1(C_2)) \\
\end{align*}
\]

Example. A Prost-enriched category is essentially a category with a preordering on each set of morphisms such that composition preserves the preordering. A Prost-enriched functor is essentially a functor \(F\) with the additional property that \(m_1 \leq m_2\) in the domain Prost-enriched category implies that \(F(m_1) \leq F(m_2)\) in the codomain Prost-enriched category. A Prost-enriched natural transformation turns out to be equivalent to just a natural transformation.

Notation. A \(M\)-enriched category is sometimes referred to as simply a \(M\)-category. A \(M\)-enriched functor is sometimes referred to as simply a \(M\)-functor. A \(M\)-enriched natural transformation is sometimes referred to as simply a \(M\)-transformation.

Definition (\(\text{CAT}(M)\)). The 2-category whose objects are \(M\)-categories, whose morphisms are \(M\)-functors, and whose 2-cells are \(M\)-transformations.

Example. The function on sets/types \(L\) can be made into a Prost-monad on the Prost-category \(\text{Rel}\).

Definition (Lax Algebra of a Prost-Monad \(\mathcal{M}\) on a Prost-Category \(C\)). A tuple \((C, a, a, i)\) whose components have the following types:
Remark. The above definition can be generalized to CAT-monads by changing \(a\) and \(i\) to be 2-cells \(\alpha\) and \(\iota\) and then imposing equations required to be satisfied by \(\alpha\) and \(\iota\).

Remark. An effector is exactly a lax algebra of the Prost-monad \(\mathbf{L}\) on the Prost-category \(\mathbf{Rel}\).

Exercise 1. Prove that there is a bijection between the set of effectors and the set of small thin multicategories (where thin means there is at most one morphism from any domain to any codomain).

Definition (Semi-strict Effector). An effector with the following additional property:

\[
\forall \bar{\epsilon}_1, \ldots, \bar{\epsilon}_n, \epsilon. \bar{\epsilon}_1 + \ldots + \bar{\epsilon}_n 
\xrightarrow{\epsilon} \epsilon \implies \exists \epsilon_1, \ldots, \epsilon_n. (\forall i. \bar{\epsilon}_i \xrightarrow{\epsilon_i}) \land [\epsilon_1, \ldots, \epsilon_n] \xrightarrow{\epsilon}
\]

Definition (Effectoid). A set \(E\) along with a unary relation \(\epsilon \mapsto \bullet\), a binary relation \(\bullet \leq \bullet\), and a ternary relation \(\bullet; \bullet \mapsto \bullet\) satisfying:

Identity

\[
\forall \epsilon, \epsilon'. \quad \exists \epsilon_\ell. \quad \epsilon \mapsto \epsilon_\ell \land \epsilon_\ell \epsilon \mapsto \epsilon'
\]

\[
\exists \epsilon_r. \quad \epsilon \mapsto \epsilon_r \land \epsilon_1 \epsilon_r \mapsto \epsilon'
\]

Associativity

\[
\forall \epsilon_1, \epsilon_2, \epsilon_3, \epsilon. \quad \exists \bar{\epsilon}. \quad \epsilon \mapsto \bar{\epsilon} \land \bar{\epsilon}_1 \bar{\epsilon}_3 \mapsto \epsilon
\]

\[
\exists \bar{\epsilon}_r. \quad \bar{\epsilon} \mapsto \bar{\epsilon}_r \land \epsilon_1 \bar{\epsilon}_r \mapsto \epsilon
\]

Reflexivity

\[
\forall \epsilon. \quad \epsilon \leq \epsilon
\]

Congruence

\[
\forall \epsilon_1, \epsilon_2, \epsilon, \epsilon'. \quad \epsilon_1 \epsilon_2 \mapsto \epsilon \land \epsilon \leq \epsilon' \implies \epsilon \mapsto \epsilon'
\]

Theorem. There is a bijection between the set of semi-strict effectors and the set of effectoids. The bijection preserves the set \(E\). The unary relation \(\epsilon \mapsto \epsilon\) corresponds to \([\epsilon] \xrightarrow{\epsilon}\); the binary relation \(\epsilon \leq \epsilon'\) corresponds to \([\epsilon] \xrightarrow{\epsilon'}\); and the ternary relation \(\epsilon_1 \epsilon_2 \mapsto \epsilon\) corresponds to \([\epsilon_1, \epsilon_2] \xrightarrow{\epsilon}\).