Effectors

Ross Tate

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Definition (Effector)**.** A tuple $\langle E, \overset{\fatsemi}{\mapsto}, \mathfrak{a}, \mathfrak{i} \rangle$ whose components have the following types:

Set of Effects E**:** `Type`

Sequence Relation $\overset{\fatsemi}{\mapsto}$**:** $\mathbb{L}(E) \times E \to$ `Prop`

Associativity $\mathfrak{a}$**:** $\forall \vec{\varepsilon}_1, \ldots, \vec{\varepsilon}_n, \varepsilon.\ \forall \varepsilon_1, \ldots, \varepsilon_n.\ (\forall i. \vec{\varepsilon}_i \overset{\fatsemi}{\mapsto} \varepsilon_i) \wedge [\varepsilon_1, \ldots, \varepsilon_n] \overset{\fatsemi}{\mapsto} \varepsilon \implies \vec{\varepsilon}_1 +\!\!+ \ldots +\!\!+ \vec{\varepsilon}_n \overset{\fatsemi}{\mapsto} \varepsilon$

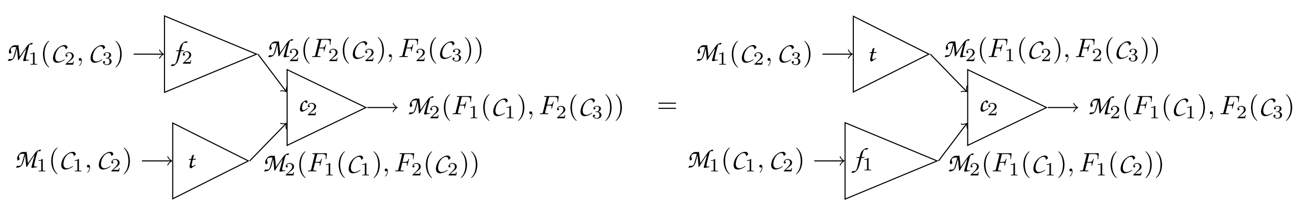
Identity $\mathfrak{i}$**:** $\forall \varepsilon.\ [\varepsilon] \overset{\fatsemi}{\mapsto} \varepsilon$

Example. An important example is where E is the singleton set and $\overset{\fatsemi}{\mapsto}$ always holds.

Definition ((Biased) **M**-Enriched Natural Transformation from $\langle F_1, f_1, \cdot, \cdot \rangle$ to $\langle F_2, f_2, \cdot, \cdot \rangle$ as **M**-enriched functors from $\langle O_1, \mathcal{M}_1, c_1, \cdot, i_1, \cdot \rangle$ to $\langle O_2, \mathcal{M}_2, c_2, \cdot, i_2, \cdot \rangle$)**.** A tuple $\langle t, \mathfrak{n} \rangle$ where the components have the following types:

Transformation t**:** For all pairs $\mathcal{C}_1, \mathcal{C}_2 : O_1$, an **M**-morphism $t : [\mathcal{M}_1(\mathcal{C}_1, \mathcal{C}_2)] \to \mathcal{M}_2(F_1(\mathcal{C}_1), F_2(\mathcal{C}_2))$

Naturality $\mathfrak{d}$**:** For all triples of objects $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 : O_1$:



Example. A **Prost**-enriched category is essentially a category with a preordering on each set of morphisms such that composition preserves the preordering. A **Prost**-enriched functor is essentially a functor F with the additional property that $m_1 \leq m_2$ in the domain **Prost**-enriched category implies that $F(m_1) \leq F(m_2)$ in the codomain **Prost**-enriched category. A **Prost**-enriched natural transformation turns out to be equivalent to just a natural transformation.

Notation. A **M**-enriched category is sometimes referred to as simply a **M**-category. A **M**-enriched functor is sometimes referred to as simply a **M**-functor. A **M**-enriched natural transformation is sometimes referred to as simply a **M**-transformation.

Definition (**CAT**(**M**))**.** The 2-category whose objects are **M**-categories, whose morphisms are **M**-functors, and whose 2-cells are **M**-transformations.

Example. The function on sets/types $\mathbb{L}$ can be made into a **Prost**-monad on the **Prost**-category **Rel**.

Definition (Lax Algebra of a **Prost**-Monad $\mathcal{M}$ on a **Prost**-Category **C**)**.** A tuple $\langle \mathcal{C}, a, \mathfrak{a}, \mathfrak{i} \rangle$ whose components have the following types:

> **Underlying Object** $\mathcal{C}$**:** $\mathbf{C}$
>
> **Operation** a**:** $M(\mathcal{C}) \to \mathcal{C}$
>
> **Associativity** $\mathfrak{a}$**:** $M(a) \,;\, a \leq \mu_{\mathcal{C}} \,;\, a : M(M(\mathcal{C})) \to \mathcal{C}$
>
> **Identity** $\mathfrak{i}$**:** $id_{\mathcal{C}} \leq \eta_{\mathcal{C}} \,;\, a : \mathcal{C} \to \mathcal{C}$

Remark. The above definition can be generalized to **CAT**-monads by changing $\mathfrak{a}$ and $\mathfrak{i}$ to be 2-cells α and ι and then imposing equations required to be satisfied by α and ι.

Remark. An effector is exactly a lax algebra of the **Prost**-monad $\mathbb{L}$ on the **Prost**-category **Rel**.

Exercise 1. Prove that there is a bijection between the set of effectors and the set of small thin multicategories (where thin means there is at most one morphism from any domain to any codomain).

Definition (Semi-strict Effector)**.** An effector with the following additional property:

$$\forall \vec{\varepsilon}_1, \ldots, \vec{\varepsilon}_n, \varepsilon.\ \vec{\varepsilon}_1 +\!\!+ \ldots +\!\!+ \vec{\varepsilon}_n \overset{\fatsemi}{\mapsto} \varepsilon \implies \exists \varepsilon_1, \ldots, \varepsilon_n.\ (\forall i. \vec{\varepsilon}_i \overset{\fatsemi}{\mapsto} \varepsilon_i) \wedge [\varepsilon_1, \ldots, \varepsilon_n] \overset{\fatsemi}{\mapsto} \varepsilon$$

Definition (Effectoid)**.** A set E along with a unary relation $\mathcal{E} \mapsto \bullet$, a binary relation $\bullet \leq \bullet$, and a ternary relation $\bullet \fatsemi \bullet \mapsto \bullet$ satisfying:

Identity $\quad \forall \varepsilon, \varepsilon'.$

$$\begin{array}{c} \exists \varepsilon_\ell.\ \mathcal{E} \mapsto \varepsilon_\ell \wedge \varepsilon_\ell \fatsemi \varepsilon \mapsto \varepsilon' \\ \Updownarrow \\ \varepsilon \leq \varepsilon' \\ \Updownarrow \\ \exists \varepsilon_r.\ \mathcal{E} \mapsto \varepsilon_r \wedge \varepsilon \fatsemi \varepsilon_r \mapsto \varepsilon' \end{array}$$

Associativity $\quad \forall \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon.$

$$\begin{array}{c} \exists \bar{\varepsilon}.\ \varepsilon_1 \fatsemi \varepsilon_2 \mapsto \bar{\varepsilon} \wedge \bar{\varepsilon} \fatsemi \varepsilon_3 \mapsto \varepsilon \\ \Updownarrow \\ \exists \hat{\varepsilon}.\ \varepsilon_2 \fatsemi \varepsilon_3 \mapsto \hat{\varepsilon} \wedge \varepsilon_1 \fatsemi \hat{\varepsilon} \mapsto \varepsilon \end{array}$$

Reflexivity $\quad \forall \varepsilon. \quad \varepsilon \leq \varepsilon$

Congruence

$$\forall \varepsilon, \varepsilon'. \quad \mathcal{E} \mapsto \varepsilon \wedge \varepsilon \leq \varepsilon' \implies \mathcal{E} \mapsto \varepsilon'$$

$$\forall \varepsilon_1, \varepsilon_2, \varepsilon, \varepsilon'. \quad \varepsilon_1 \fatsemi \varepsilon_2 \mapsto \varepsilon \wedge \varepsilon \leq \varepsilon' \implies \varepsilon_1 \fatsemi \varepsilon_2 \mapsto \varepsilon'$$

Theorem. *There is a bijection between the set of semi-strict effectors and the set of effectoids. The bijection preserves the set E. The unary relation $\mathcal{E} \mapsto \varepsilon$ corresponds to $[\,] \overset{\fatsemi}{\mapsto} \varepsilon$; the binary relation $\varepsilon \leq \varepsilon'$ corresponds to $[\varepsilon] \overset{\fatsemi}{\mapsto} \varepsilon'$; and the ternary relation $\varepsilon_1 \fatsemi \varepsilon_2 \mapsto \varepsilon$ corresponds to $[\varepsilon_1, \varepsilon_2] \overset{\fatsemi}{\mapsto} \varepsilon$.*