Colimits

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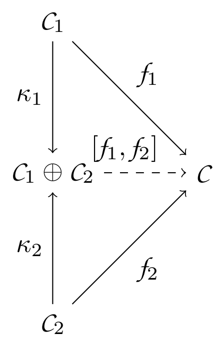
October 20, 2014

Definition (Coproduct of $\mathcal{C}_1$ and $\mathcal{C}_2$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ are objects of $\mathbf{C}$)**.** An object, denoted $\mathcal{C}_1 \oplus \mathcal{C}_2$ (although more traditionally with $\mathcal{C}_1 + \mathcal{C}_2$), along with morphisms $\kappa_1 : \mathcal{C}_1 \to \mathcal{C}_1 \oplus \mathcal{C}_2$ and $\kappa_2 : \mathcal{C}_2 \to \mathcal{C}_1 \oplus \mathcal{C}_2$ with the property that, for any object $\mathcal{C}$ and morphisms $f_1 : \mathcal{C}_1 \to \mathcal{C}$ and $f_2 : \mathcal{C}_2 \to \mathcal{C}$, there exists a unique morphism, denoted $[f_1, f_2]$, making the following diagram commute:



Example. In **Set**, $A \oplus B$ is the disjoint union $A + B$ of A and B. In **Mon**, $\mathcal{A} \oplus \mathcal{B}$ is the set of alternating lists of A and B non-identity elements with a variant of concatonenation as its multiplication. In **Rel**(2), the coproduct of $\langle A, \sqsubset_1 \rangle$ and $\langle B, \sqsubset_2 \rangle$ is the disjoint union of the two sets where left elements are related by $\sqsubset_1$ and right elements are related by $\sqsubset_2$ and no left and right elements are related to each other. In **Cat**, $\mathbf{A} \otimes \mathbf{B}$ uses the disjoint union of the objects and uses alternating paths for morphisms. In **Rel**, the disjoint union of A and B is the coproduct of A and B.

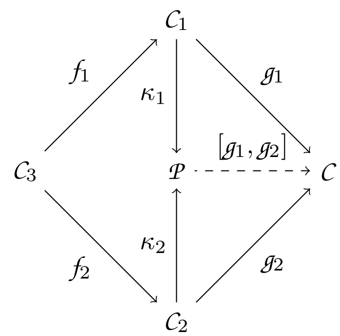
Definition (Initial Object of $\mathbf{C}$)**.** An object, denoted 0, with the property that, for any object $\mathcal{C}$, there exists a unique morphism, denoted $[]$, from 0 to $\mathcal{C}$.

Example. In **Set**, any empty set is an initial object. In **Mon**, any singleton monoid is an initial monoid. In **Rel**(2), $\langle \mathbb{0}, \bot \rangle$ is the initial binary relation. In **Cat**, any category with no objects is an initial category. In **Rel**, any empty set is an initial object.

Definition (Coequalizer of morphisms $f_1, f_2 : \mathcal{C}_1 \to \mathcal{C}_2$)**.** An object $\mathcal{E}$ along with a morphism $\kappa : \mathcal{C}_1 \to \mathcal{E}$ such that $f_1 ; \kappa = f_2 ; \kappa$ and with the property that, for any other object $\mathcal{C}$ and morphism $f : \mathcal{C}_1 \to \mathcal{C}$ such that $f_1 ; f = f_2 ; f$, there exists a unique morphism $[f] : \mathcal{E} \to \mathcal{C}$ such that $[f] ; \kappa = f$.

Example. In **Set**, the coequalizer of functions $f_1, f_2 : X \to Y$ is the set $\frac{Y}{\approx}$ where $y_1 \approx y_2$ is defined as $\exists x.\ f_1(x) = y_1 \wedge f_2(x) = y_2$. In **Mon**, one uses the above construction except furthermore requires $\approx$ to satisfy $\forall y_1, y_1', y_2, y_2'.\ y_1 \approx y_1' \wedge y_2 \approx y_2' \Rightarrow y_1 * y_2 \approx y_1' * y_2'$. In **Rel**(2), the coequalizer of functions $f_1, f_2 : X \to Y$ is the set $\frac{Y}{\approx}$ where $y_1 \approx y_2$ is defined as $\exists x.\ f_1(x) = y_1 \wedge f_2(x) = y_2$, and two equivalence classes are related if any of their elements are related. In **Cat**, one builds the coequalizer for the components on objects and then combines the above techniques to build equivalence classes of morphisms. **Rel** does not have coequalizers for some pairs of binary relations.

Definition (Pushout of morhisms $f_1 : \mathcal{C}_3 \to \mathcal{C}_1$ and $f_2 : \mathcal{C}_3 \to \mathcal{C}_2$)**.** An object $\mathcal{P}$ along with morphisms $\kappa_1 : \mathcal{C}_1 \to \mathcal{P}$ and $\kappa_2 : \mathcal{C}_2 \to \mathcal{P}$ such that $f_1 ; \kappa_1 = f_2 ; \kappa_2$ and with the property that, for any object $\mathcal{C}$ and morphisms $g_1 : \mathcal{C}_1 \to \mathcal{C}$ and $g_2 : \mathcal{C}_2 \to \mathcal{C}$ such that $f_1 ; g_1 = f_2 ; g_1$, there exists a unique morphism, denoted $[g_1, g_2]$, making the following diagram commute:



Example. In **Set**, the pushout of functions $f_1 : Z \to X$ and $f_2 : Z \to Y$ is the set $\frac{X+Y}{\approx}$, where $\approx$ is the weakest equivalence such that $\forall z : Z.\ \texttt{inl}(f_1(z)) \approx \texttt{inr}(f_2(z))$, along with the obvious coprojection functions.

Exercise 1. Note that the construction of pushouts in **Set** is built from a coproduct and a coequalizer. Prove that if a category has coproducts for all objects and equalizers for all parallel morphism pairs, then it has pullbacks for all morphism pairs with the same codomain.

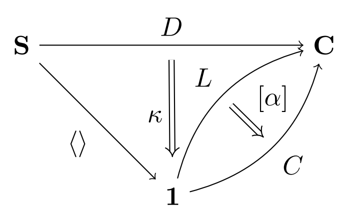
Definition (Colimit of a functor $D : \mathbf{S} \to \mathbf{C}$)**.** An object L of **C** along with a natural transformation $\kappa : D \Rightarrow L$ with the property that, for any object C and natural transformation $\alpha : D \Rightarrow C$, there exists a unique morphism $[\alpha] : L \to C$ such that $\kappa\,;[\alpha]$ equals α.

Definition (Scheme and Diagram)**.** Given $D : \mathbf{S} \to \mathbf{C}$, the category **S** is called the scheme and the functor D is called the diagram in **C**.

Example. Coproducts correspond to colimits of diagrams with scheme $\mathbb{2}$, the category with 2 objects and only identity morphisms. Initial objects correspond to colimits of the diagram with scheme $\mathbf{0}$, the category with no objects or morphisms. Coequalizers correspond to colimits of diagrams with the scheme $\bullet_1 \rightrightarrows \bullet_2$. Pushouts correspond to colimits of diagrams with the scheme $\bullet_1 \leftarrow \bullet_3 \to \bullet_2$.

Exercise 2. Prove that a category has colimits for all diagrams with scheme **S** if and only if the functor Δ from **C** to $\mathbf{S} \multimap \mathbf{C}$, mapping each object to its corresponding constant functor and each morphism to its corresponding constant natural transformation, has a left adjoint.

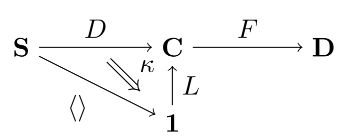
Remark. Given a functor $D : \mathbf{S} \to \mathbf{C}$, a colimit is a functor $L : \mathbf{1} \to \mathbf{C}$ and natural transformation $\kappa : D \Rightarrow \langle\rangle_{\mathbf{S}}\,; L$ with the property that, for any functor $C : \mathbf{1} \to \mathbf{C}$ and natural transformation $\alpha : D \Rightarrow \langle\rangle_{\mathbf{S}}\,; L$, there exists a unique natural transformation $[\alpha] : L \Rightarrow C$ such that the natural transformation specified in the following diagram equals α:



Definition (Finitely Cocomplete)**.** A category that has a colimit for all diagrams with finite schemes, meaning the scheme has a finite set of objects and morphisms.

Exercise 3. Prove that a category is finitely cocomplete if and only if it has a initial objects, coproducts, and coequalizers.

Definition (Preserves **S**-Colimits)**.** A functor $F : \mathbf{C} \to \mathbf{D}$ with the property that, for any D, L, and κ, if $L : \mathbf{1} \to \mathbf{C}$ and $\kappa : D \Rightarrow \langle\rangle\,; L$ is a colimit of $D : \mathbf{S} \to \mathbf{C}$, then $L\,; F$ and the following natural transforation is a colimit of $D\,; F$:



Definition ((Finitely) Cocontinuous)**.** A functor that preserves all colimits is called *cocontinuous.* A functor that preserves all finite colimits is called *finitely* cocontinuous.

Exercise 4. Prove that every left-adjoint functor is cocontinuous.