Categories

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Definition ((Biased) (**Set**-enriched) Category)**.** A tuple $\langle O, M, ;, \mathfrak{a}, id, \mathfrak{i}\rangle$ where the components have the following types:

Objects O**:** $\mathtt{Type}_1$

Morphisms M**:** $O \times O \to \mathtt{Type}$

Composition ;**:** $\forall \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 : O.\ M(\mathcal{C}_1, \mathcal{C}_2) \times M(\mathcal{C}_2, \mathcal{C}_3) \to M(\mathcal{C}_1, \mathcal{C}_3)$ (infix — objects implicit)

Associativity $\mathfrak{a}$**:** $\forall \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4 : O.\ \forall m_1 : M(\mathcal{C}_1, \mathcal{C}_2), m_2 : M(\mathcal{C}_2, \mathcal{C}_3), m_3 : M(\mathcal{C}_3, \mathcal{C}_4).\ (m_1 ; m_2) ; m_3 = m_1 ;(m_2 ; m_3)$

Identities id**:** $\forall \mathcal{C} : O.\ M(\mathcal{C}, \mathcal{C})$ (object implicit)

Identity $\mathfrak{i}$**:** $\forall \mathcal{C}_1, \mathcal{C}_2 : O, m : M(\mathcal{C}_1, \mathcal{C}_2).\ id ; m = m = m ; id$

Notation. When the category is implicit from context, we use $\mathcal{C}_1 \to \mathcal{C}_2$ to denote the type $M(\mathcal{C}_1, \mathcal{C}_2)$, referred to as morphisms from $\mathcal{C}_1$ to $\mathcal{C}_2$.

Notation. When the category is implicit from context, we use $\mathcal{C}_1 \xrightarrow{m} \mathcal{C}_2$ to denote $m : \mathcal{C}_1 \to \mathcal{C}_2$.

Definition (Domain,Codomain,Source,Target)**.** Given a morphism $\mathcal{C}_1 \xrightarrow{m} \mathcal{C}_2$, we refer to $\mathcal{C}_1$ as the domain (or source) of m, and to $\mathcal{C}_2$ as the codomain (or target) of m.

Notation. We use $\mathcal{C}_1 \xrightarrow{m_1} \mathcal{C}_2 \xrightarrow{m_2} \mathcal{C}_2$ (and longer chains) to denote $m_1 ; m_2$.

Notation. We use $m_2 \circ m_1$ to denote $m_1 ; m_2$.

Example. $\mathbf{Set} = \langle \mathtt{Type}, \lambda\langle \tau_1, \tau_2\rangle.\ \tau_1 \to \tau_2, \lambda\langle \tau_1, \tau_2, \tau_3\rangle.\ \lambda\langle f, g\rangle.\ \lambda x.\ g(f(x)), \centerdot, \lambda\tau.\ \lambda x.\ x, \centerdot\rangle$

$\mathbf{n} = \langle \mathbb{n}, \leq, \mathfrak{transitivity}, \mathfrak{proof\text{-}irrelevance}, \mathfrak{reflexivity}, \mathfrak{proof\text{-}irrelevance}\rangle$ where $\mathbb{n}$ is $\{1, \ldots, n\}$

$\omega = \langle \mathbb{N}, \leq, \mathfrak{transitivity}, \mathfrak{proof\text{-}irrelevance}, \mathfrak{reflexivity}, \mathfrak{proof\text{-}irrelevance}\rangle$

$\mathbf{Rel} = \langle \mathtt{Type}, \lambda\langle \tau_1, \tau_2\rangle.\ \tau_1 \times \tau_2 \to \mathtt{Prop}, \lambda\langle \tau_1, \tau_2, \tau_3\rangle.\ \lambda\langle \phi_1, \phi_2\rangle.\ \lambda\langle t_1, t_3\rangle.\ \exists t_2.\ \phi_1(t_1, t_2) \wedge \phi_2(t_2, t_3), \centerdot, \lambda\langle t, t'\rangle.\ t = t', \centerdot\rangle$

$$\mathbf{Prost} \left\langle \begin{array}{l} \sum_{\tau:\mathtt{Type}} \sum_{R:\tau\times\tau\to\mathtt{Prop}} (\forall t : \tau.\ R(t, t)) \wedge (\forall t_1, t_2, t_3 : \tau.\ R(t_1, t_2) \times R(t_2, t_3) \to R(t_1, t_3)), \\ \lambda\langle\langle \tau_1, R_1, \centerdot\rangle, \langle \tau_2, R_2, \centerdot\rangle\rangle.\ \sum_{f:\tau_1\to\tau_2} \forall t_1, t_2 : \tau_1.\ R_1(t_1, t_2) \to R_2(f(t_1), f(t_2)), \\ \lambda\langle \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3\rangle.\ \lambda\langle\langle f_1, \centerdot\rangle, \langle f_2, \centerdot\rangle\rangle.\ \langle \lambda x.\ f_2(f_1(x)), \centerdot\rangle, \centerdot, \lambda\langle \tau, R, \centerdot\rangle.\ \langle \lambda x.\ x, \centerdot\rangle, \centerdot \end{array} \right\rangle$$

$\mathbf{Mat} = \langle \mathbb{N}, \lambda\langle n_1, n_2\rangle.\ \mathbb{R}^{n_2\times n_1}, \lambda\langle n_1, n_2, n_3\rangle.\ \lambda\langle M_1, M_2\rangle.\ M_2 \cdot M_1, \centerdot, \lambda n.\ \delta^n_n, \centerdot\rangle$

$\mathbb{\Delta}_n = \langle \mathbb{n}, \lambda\langle n_1, n_2\rangle.\ \{\sigma : \mathbb{n}_1 \to \mathbb{n}_2 \mid \forall n_1, n_2 : \mathbb{n}_1.\ n_1 \leq n_2 \Rightarrow \sigma n_1 \leq \sigma n_2\}, \lambda\langle n_1, n_2, n_3\rangle.\ \lambda\langle \sigma_1, \sigma_2\rangle.\ \sigma_1 ; \sigma_2, \centerdot, \lambda n.\ \lambda x.\ x, \centerdot\rangle$

$\mathbb{\Delta} = \langle \mathbb{N}, \lambda\langle n_1, n_2\rangle.\ \{\sigma : \mathbb{n}_1 \to \mathbb{n}_2 \mid \forall n_1, n_2 : \mathbb{n}_1.\ n_1 \leq n_2 \Rightarrow \sigma n_1 \leq \sigma n_2\}, \lambda\langle n_1, n_2, n_3\rangle.\ \lambda\langle \sigma_1, \sigma_2\rangle.\ \sigma_1 ; \sigma_2, \centerdot, \lambda n.\ \lambda x.\ x, \centerdot\rangle$

$$\mathbf{Sig} = \left\langle \begin{array}{l} \sum_{O:\mathtt{Type}} O \to \mathtt{Type}, \\ \lambda\langle\langle O_1, N_1\rangle, \langle O_2, N_2\rangle\rangle.\ \sum_{f:O_1\to O_2} \prod_{o:O_1} N_2(f(o)) \to N_1(o), \\ \lambda\langle\langle O_1, N_1\rangle, \langle O_2, N_2\rangle, \langle O_3, N_3\rangle\rangle.\ \lambda\langle\langle f_1, n_1\rangle, \langle f_2, n_2\rangle\rangle.\ \langle \lambda o.\ f_2(f_1(o)), \lambda o.\ \lambda n.\ n_1(o)(n_2(f_1(o))(n))\rangle, \centerdot, \\ \lambda\langle O, N\rangle.\ \langle \lambda o.\ o, \lambda o.\ \lambda n.\ n\rangle, \centerdot \end{array} \right\rangle$$

$$\mathbf{Alg}(\Omega : \mathbf{Sig}) = \left\langle \begin{array}{l} \sum_{A:\mathtt{Type}} \prod_{op\mapsto N\in\Omega} (N \to A) \to A, \\ \lambda\langle\langle A, a\rangle, \langle B, b\rangle\rangle.\ \sum_{f:A\to B} \forall op \mapsto N \in \Omega.\ \forall i : N \to A.\ b_{op}(\lambda n.\ f(i(n))) = f(a_{op}(i)), \\ \lambda\langle \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\rangle.\ \lambda\langle\langle f_1, \centerdot\rangle, \langle f_2, \centerdot\rangle\rangle.\ \langle \lambda x.\ f_2(f_1(x)), \centerdot\rangle, \centerdot, \lambda\langle A, a\rangle.\ \langle \lambda x.\ x, \centerdot\rangle, \centerdot \end{array} \right\rangle$$

$$\mathbf{Rel}(\Phi : \mathbf{Sig}) = \left\langle \begin{array}{l} \sum_{A:\mathtt{Type}} \prod_{rel\mapsto N\in\Phi} (N \to A) \to \mathtt{Prop}, \\ \lambda\langle\langle R, \phi\rangle, \langle S, \psi\rangle\rangle.\ \sum_{f:R\to S} \forall rel \mapsto N \in \Phi.\ \forall i : N \to A.\ \phi_{rel}(i) \implies \psi_{rel}(\lambda n.\ f(i(n))), \\ \lambda\langle \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3\rangle.\ \lambda\langle\langle f_1, \centerdot\rangle, \langle f_2, \centerdot\rangle\rangle.\ \langle \lambda x.\ f_2(f_1(x)), \centerdot\rangle, \centerdot, \lambda\langle A, a\rangle.\ \langle \lambda x.\ x, \centerdot\rangle, \centerdot \end{array} \right\rangle$$

Exercise 1. Define a category $\mathbf{Mon}_b$ with biased monoids as its objects and biased monoid homomorphisms as its morphisms.

Exercise 2. Define a category $\mathbf{Mon}_u$ with unbiased monoids as its objects and unbiased monoid homomorphisms as its morphisms.