Monad Algebras

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Definition (Algebra of a **CAT**-Monad $\langle \mathbf{C}, M, \mu, \mathbf{.}, \eta, \mathbf{.} \rangle$)**.** A tuple $\langle \mathcal{C}, a, \mathfrak{a}, \mathfrak{i} \rangle$ of the following form:

> **Underlying Object** $\mathcal{C}$**:** $\mathbf{C}$
>
> **Operation** a**:** $M(\mathcal{C}) \to \mathcal{C}$
>
> **Associativity** $\mathfrak{a}$**:** $M(a) \,;\, a = \mu_{\mathcal{C}} \,;\, a : M(M(\mathcal{C})) \to \mathcal{C}$
>
> **Identity** $\mathfrak{i}$**:** $id_{\mathcal{C}} = \eta_{\mathcal{C}} \,;\, a : \mathcal{C} \to \mathcal{C}$

Remark. The above definition is also known as an Eilenberg-Moore algebra.

Example. The algebras for $\mathbb{L}$ are the (unbiased) monoids. The algebras for $\mathbb{M}$ are the (unbiased) commutative monoids. The algebras for $\mathbb{S}$ are the (unbiased) idempotent commutative monoids.

Definition (Morphism of Monad Algebras from $\langle \mathcal{C}_1, a_1, \mathbf{.}, \mathbf{.} \rangle$ to $\langle \mathcal{C}_2, a_2, \mathbf{.}, \mathbf{.} \rangle$)**.** A tuple $\langle f, \mathfrak{d} \rangle$ where f is a morphism from $\mathcal{C}_1$ to $\mathcal{C}_2$ and $\mathfrak{d}$ is a proof that $M(f) \,;\, a_2$ equals $a_1 \,;\, f$.

Example. Just like how an algebra for $\mathbb{L}$ corresponds to a monoid, a morphism of $\mathbb{L}$-algebras corresponds to a monoid homomorphism.

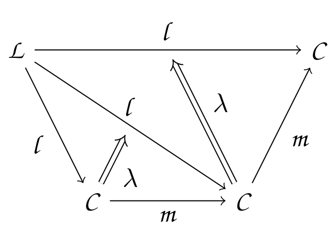
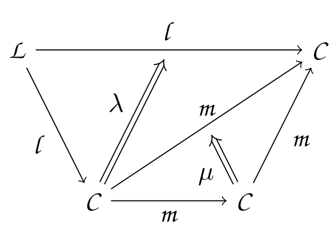
Definition ($\mathbf{Alg}(\mathcal{M})$ where $\mathcal{M}$ is a **CAT**-Monad)**.** The category whose objects are $\mathcal{M}$-algebras and whose morphisms are $\mathcal{M}$-algebra morphisms. Identities and composition of morphisms are inherited from $\mathbf{C}$.

Remark. $\mathbf{Alg}(\mathcal{M})$ is known as the Eilenberg-Moore category of $\mathcal{M}$.

Example. Abusing notation, $\mathbf{Alg}(\mathbb{L})$ is $\mathbf{Mon}_{Unbiased}$, and $\mathbf{Alg}(\mathbb{M})$ is $\mathbf{CommMon}_{Unbiased}$.

Exercise 1. Show that a monad morphism from $\mathcal{M}_1$ to $\mathcal{M}_2$ provides a functor from $\mathbf{Alg}(\mathcal{M}_2)$ to $\mathbf{Alg}(\mathcal{M}_1)$.

Definition ($\langle \mathcal{C}, m, \mu, \mathbf{.}, \eta, \mathbf{.} \rangle$-Premodule in a 2-Category $\mathbf{C}$)**.** A tuple $\langle \mathcal{L}, \ell, \lambda, \mathfrak{d}, \mathfrak{i} \rangle$ of the following form:

> **Object** $\mathcal{L}$**:** $\mathbf{C}$
>
> **Morphism** ℓ**:** $\mathcal{L} \to \mathcal{C}$
>
> **Action** λ**:** $\ell \,;\, m \Rightarrow \ell$
>
> **Distributivity** $\mathfrak{d}$**:** A proof that  equals  .
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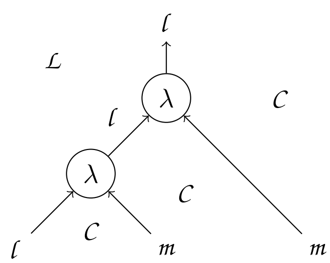
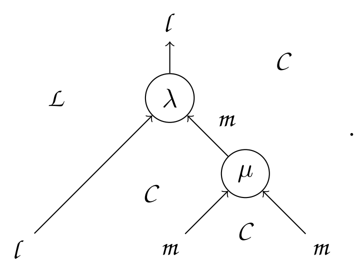
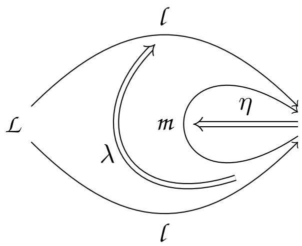
> In other words,  equals  .

Identity $\mathfrak{i}$: A proof that



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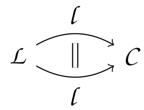

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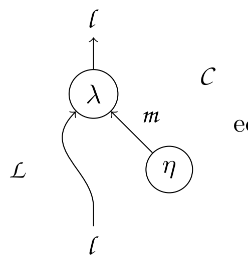

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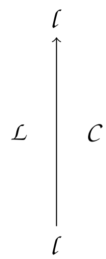

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Remark. A premodule is more commonly called a left module.

Remark. An algebra for a **CAT**-monad is simply a premodule where $\mathcal{L}$ is **1**.

Theorem. *For every monad $\langle \mathcal{C}, m, \mu, \mathfrak{d}, \eta, \mathfrak{i} \rangle$, the tuple $\langle \mathcal{C}, m, \mu, \mathfrak{d}, \mathfrak{i} \rangle$ is a premodule of that monad.*

Example. Suppose we have a **CAT**-monad $\mathcal{M}$ whose components are $\langle \mathbf{C}, M, \mu, \bullet, \eta, \bullet \rangle$. Let U be the functor from $\mathbf{Alg}(\mathcal{M})$ to $\mathbf{C}$ that maps each algebra $\langle \mathcal{C}, a, \bullet, \bullet \rangle$ to $\mathcal{C}$ and each algebra morphism $\langle f, \bullet \rangle$ to f. Let $\alpha : U \,; M \Rightarrow M$ be the natural transformation mapping each algebra $\langle \mathcal{C}, a, \bullet, \bullet \rangle$ to the morphism $a : M(U(\langle \mathcal{C}, a, \bullet, \bullet \rangle)) = M(\mathcal{C}) \to \mathcal{C}$. This forms a $\mathcal{M}$-premodule: α distributes and preserves identity because each operation a is associative and preserves identity.

Exercise 2. Prove that for any **CAT**-monad $\mathcal{M}$ and $\mathcal{M}$-premodule $\langle \mathbf{L}, L, \lambda, \bullet, \bullet \rangle$, there is a unique functor $L' : \mathbf{L} \to \mathbf{Alg}(\mathcal{M})$ such that $L = L' \,; U$ and $\lambda = L' \cdot \alpha$.

Remark. Given a 2-category $\mathbf{C}$, one can construct an opetory with the same 0-cells and 1-cells and with a 2-cell for each 2-cell from the composition of the inputs to the output. **1** is the opetory with one 0-cell $\mathcal{C}$, one 1-cell $m : \mathcal{C} \to \mathcal{C}$, and one 2-cell from $m^n \Rightarrow m$ for each $n : \mathbb{N}$. A monad $\mathcal{M}$ in $\mathbf{C}$ corresponds to a functor M of opetories from **1** to $\mathbf{C}$. Let $\mathbf{1}_\ell$ be the operatory with two 0-cells $\mathcal{L}$ and $\mathcal{C}$, two 1-cells $\ell : \mathcal{L} \to \mathcal{C}$ and $m : \mathcal{C} \to \mathcal{C}$, and one 2-cell from ℓm^n to ℓ for each $n : \mathbb{N}$ and one 2-cell from $m^n \Rightarrow m$ for each $n : \mathbb{N}$. There is a unique functor of opetories from **1** to $\mathbf{1}_\ell$, which we will call I_ℓ. An $\mathcal{M}$-premodule $\mathcal{L}$, then, corresponds to a functor L of opetories from $\mathbf{1}_\ell$ to $\mathbf{C}$ such that $I_\ell \,; L$ equals M.

Exercise 3. Show that a monad morphism from $\mathcal{M}_1$ to $\mathcal{M}_2$ provides a functor from $\mathbf{Alg}(\mathcal{M}_2)$ to $\mathbf{Alg}(\mathcal{M}_1)$.

Remark. The monad morphism from the $\mathbb{L}$ monad to the $\mathbb{M}$ monad corresponds to the inclusion functor from **CommMon** to **Mon**.