**Definition (Unbiased Monoid).** A tuple \( \langle M, \Pi, a, i \rangle \) where the components have the following types:

- **Underlying Set** \( M \): Type
- **Aggregator** \( \Pi \): \( L M \to M \)
- **Associativity** \( a \): \( \forall n : \mathbb{N}, \vec{m}_1, \ldots, \vec{m}_n : L M. \Pi [\Pi \vec{m}_1, \ldots, \Pi \vec{m}_n] = \Pi (\vec{m}_1 + + \ldots + + \vec{m}_n) \)
- **Identity** \( i \): \( \forall m : M. m = \Pi [m] \)

**Notation.** We use \( m_1 * \cdots * m_n \) to denote \( \prod [m_1, \ldots, m_n] \).

**Remark.** This definition provides an \( n \)-ary operator for every \( n \), which is why we call it *unbiased*. The former definition provided an operator for only arities 0 and 2, which is why we call it *biased*.

**Example.**
- \( \mathbb{N}_\Sigma = \langle \mathbb{N}, \Sigma, *, * \rangle \)
- \( \mathbb{N}_\Pi = \langle \mathbb{N}, \Pi, *, * \rangle \)
- \( \mathbb{N}_{max} = \langle \mathbb{N}, \text{max}, *, * \rangle \)
- \( \mathbb{Z}_\Sigma = \langle \mathbb{Z}, \Sigma, *, * \rangle \)
- \( \mathbb{Z}_\Pi = \langle \mathbb{Z}, \Pi, *, * \rangle \)
- \( \mathbb{R}_\Sigma = \langle \mathbb{R}, \Sigma, *, * \rangle \)
- \( \mathbb{R}_\Pi = \langle \mathbb{R}, \Pi, *, * \rangle \)
- \( \mathbb{N}_\Sigma^\infty = \langle \mathbb{N}^\infty, \Sigma, *, * \rangle \) where \( \forall \vec{n} : L (\mathbb{N}^\infty). \in \vec{n} \implies \Sigma \vec{n} = \infty \)
- \( \mathbb{N}_{min} = \langle \mathbb{N}^\infty, \text{min}, *, * \rangle \) where \( \forall \vec{n} : L (\mathbb{N}^\infty). \text{min} \vec{n} = \infty \implies \vec{n} = [\infty, \ldots, \infty] \) (including \([ \ ]\) )
- \( \mathbb{B}_\wedge = \langle \mathbb{B}, \wedge, *, * \rangle \)
- \( \mathbb{B}_\vee = \langle \mathbb{B}, \vee, *, * \rangle \)
- \( (L T)_\Sigma = \langle L T, \Sigma, *, * \rangle \) where \( \Sigma [\vec{t}_1, \ldots, \vec{t}_n] = \vec{t}_1 + + \ldots + + \vec{t}_n \) ([ ] when \( n = 0 \))
- \( (M T)_\Sigma = \langle M T, \Sigma, *, * \rangle \)
- \( (S T)_\cup = \langle S T, \cup, *, * \rangle \)
- \( (P T)_\cup = \langle P T, \cup, *, * \rangle \)
- \( (P T)_\cap = \langle P T, \cap, *, * \rangle \)

**Exercise 1.** Give a bijection between unbiased monoids and biased monoids.

**Remark.** The mapping of underlying sets and operations is fairly straightforward, but proving the associativity and identity laws is challenging, especially in the biased-to-unbiased direction.

**Notation.** We call the function from unbiased monoids to biased monoids \( \text{Bias} \), and the inverse \( \text{Unbias} \).