Adjunctions
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**Definition** (Adjunction for a 2-Category $\mathcal{C}$). A tuple $\langle \mathcal{C}, \mathcal{D}, f, g, \eta, \varepsilon, f, g \rangle$ whose components have the following types:

- $\mathcal{C}$ is an object of $\mathcal{C}$
- $\mathcal{D}$ is an object of $\mathcal{C}$
- $f$ is a morphism of $\mathcal{C}$ from $\mathcal{C}$ to $\mathcal{D}$
- $g$ is a morphism of $\mathcal{C}$ from $\mathcal{D}$ to $\mathcal{C}$
- $\eta$ is a 2-cell of $\mathcal{C}$ from $\mathcal{C}$ to $f$;
- $\varepsilon$ is a 2-cell of $\mathcal{C}$ from $g$; $f$ to $\mathcal{D}$

$f$ is a proof that $f$ equals $f$. In other words, $\xymatrix{ \mathcal{C} \ar[r]^f & \mathcal{D} \ar[l]_{\eta} }$ equals $\xymatrix{ \mathcal{C} \ar[r]^f & \mathcal{D} }$.

$g$ is a proof that $g$ equals $g$. In other words, $\xymatrix{ \mathcal{D} \ar[r]^g & \mathcal{C} \ar[l]_{\varepsilon} }$ equals $\xymatrix{ \mathcal{D} \ar[r]^g & \mathcal{C} }$.

**Definition** (Left/Right Adjoint). $f$ above is called the left adjoint, and $g$ above is called the right adjoint. A morphism of a 2-category is a left/right adjoint if it is the left/right adjoint of some adjunction.

**Exercise 1.** Prove that there is a bijection between adjunctions in $\text{Cat}$ and adjunctions via transpositions.

**Example.** Consider $\text{Prost}$. Suppose we had a pair of preordered sets $\langle \mathcal{C}, \leq \rangle$ and $\langle \mathcal{D}, \leq \rangle$, and we want to make an adjunction out of some relation-preserving functions $f : \mathcal{C} \to \mathcal{D}$ and $g : \mathcal{D} \to \mathcal{C}$. Then $\eta$ exists if and only if $\forall c : \mathcal{C}. \ c \leq g(f(c))$, and $\beta$ exists if and only if $\forall d : \mathcal{D}. \ f(g(d)) \leq g$. If $\eta$ and $\beta$ exist, then $f$ and $g$ are trivial since $\text{Prost}$ is a locally thin 2-category. Such a situation is called a monotone Galois connection.