

Multicategories

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Remark. Because of the nature of the problem space of this assignment (e.g. n -ary inputs), you will need to develop a balance between formality and clarity/brevity. For example, carefully select examples that are simple enough to reason about concisely but complex enough that this reasoning will clearly generalize to all other cases. Also, explicitly state how given properties of one component (e.g. the domain of a functor) imply required properties of another component (e.g. the codomain of a functor), possibly explicitly illustrating this connection with an example. Do not get buried in the irrelevant details.

Exercise 1. Prove that there is a multicategory **Mon** whose objects are monoids and whose morphisms are the multilinear homomorphisms with composition and identity are inherited from **Set**. This amounts to defining what nullary multilinear homomorphisms are, and proving that multilinear homomorphisms are closed under composition and identity.

Definition ((Unbiased) (**Set**-Enriched) Category). A tuple $\langle O, M, \cdot, \mathbf{a}, \mathbf{i} \rangle$ whose components have the following types:

Objects O : Type_1

Morphisms M : $O \times O \rightarrow \text{Type}$

Composition \cdot : Maps each path of morphisms from object C_1 to object C_2 to a morphism from C_1 to C_2

Associativity \mathbf{a} : Given a path of paths, composing each subpath into a morphism and then composing the resulting path of morphisms must produce the same result as composing the path resulting from flattening the path of paths.

Identity \mathbf{i} : Composing the singleton path results in the contained morphism.

Theorem. *There is a bijection between biased and unbiased categories.*

Proof. The proof is essentially the same as that for the bijection between biased and unbiased monoids. \square

Exercise 2. Prove that there is a bijection between the set of categories and the set of pairs $\langle O, M \rangle$ where O is an element of Type_1 and M is a functor of multicategories from $\text{Path}(O)$ to **Set**.

Exercise 3. Give an example of an internal monoid of **Mon** whose underlying set is \mathbb{N} .

Exercise 4. Define a multicategory **M** with the property that, for any multicategory **C**, there is a bijection between the set of functors from **M** to **C** and the set of internal monoids of **C**.

Definition. A commutative monoid is a monoid **M** with the property $\forall m_1, m_2 : M. m_1 * m_2 = m_2 * m_1$. **CommMon** is the multicategory whose objects are commutative monoids and whose morphisms are multilinear monoid homomorphisms.

Exercise 5. Prove that the category **CommMon** can be enriched in the multicategory **CommMon**. That is, show that there is a functor of multicategories from $\text{Path}(O_{\text{CommMon}})$ to **CommMon** that when composed with the underlying functor of multicategories from **CommMon** to **Set** produces the functor of multicategories from $\text{Path}(O_{\text{CommMon}})$ to **Set** defining the category **CommMon**.