Monoids

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Exercise 1. Given monoids $\mathcal{A}$ and $\mathcal{B}$, give a monoidal structure $\mathcal{A} \,\&\, \mathcal{B}$ to the set $A \times B$ such that the projection functions π_A and π_B are monoid homomorphisms from $\mathcal{A} \,\&\, \mathcal{B}$ to $\mathcal{A}$ and $\mathcal{B}$ respectively.

Notation. $\mathcal{A} \,\&\, \mathcal{B}$ is called the *product* of $\mathcal{A}$ and $\mathcal{B}$, though it is more commonly denoted as $\mathcal{A} \times \mathcal{B}$ and sometimes called the *direct* product.

Exercise 2. Determine the monoid "$\top$" with the property that for every monoid $\mathcal{A}$ there is exactly one monoid homomorphism from $\mathcal{A}$ to $\top$.

Exercise 3. Determine the monoid "0" with the property that for every monoid $\mathcal{A}$ there is exactly one monoid homomorphism from 0 to $\mathcal{A}$.

Remark. $\top$ is called the *terminal* monoid (more commonly denoted with 1), and 0 is called the *initial* monoid.

Definition. A multilinear homomorphism from $\mathcal{A}$ and $\mathcal{B}$ to $\mathcal{C}$ is a function $f : A \times B \to C$ such that $f(e_{\mathcal{A}}, b) = e_{\mathcal{C}}$ always, $f(a_1 * a_2, b) = f(a_1, b) * f(a_2, b)$ always, $f(a, e_{\mathcal{B}}) = e_{\mathcal{C}}$ always, and $f(a, b_1 * b_2) = f(a, b_1) * f(a, b_2)$ always. In other words, fixing either argument produces a monoid homomorphism.

Definition. Given a type τ and a binary relation $\approx : \tau \times \tau \to \texttt{Prop}$, the type $\frac{\tau}{\approx}$ is called the quotient. Set theoretically, it is the set of all equivalence classes of $\approx$ on τ. There is a function $\lambda t.\ \frac{t}{\approx} : \tau \to \frac{\tau}{\approx}$ mapping each element of τ to its equivalence class. To construct functions from $\frac{\tau}{\approx}$ to another type τ', one uses $\texttt{select}\ t\ \texttt{from}\ q\ \texttt{in}\ e[t]\ \texttt{using}\ \mathfrak{p}$, where q is a $\frac{\tau}{\approx}$, t is a variable bound to some τ value in q, $e[t]$ is an expression of type τ' indicating how to use t, and $\mathfrak{p}$ is a proof that $\forall t, t' : \tau.\ t \approx t' \Rightarrow e[t] = e[t']$.

Definition. Given monoids $\mathcal{A}$ and $\mathcal{B}$, define the equivalence relation $\approx$ on $\mathbb{L}(A \times B)$ to be the least equivalence relation such that:

1. $\forall \vec{m}_1, \vec{m}_1', \vec{m}_2, \vec{m}_2' : \mathbb{L}(A \times B).\ \vec{m}_1 \approx \vec{m}_1' \wedge \vec{m}_2 \approx \vec{m}_2' \implies \vec{m}_1 \mathbin{+\!\!+} \vec{m}_2 \approx \vec{m}_1' \mathbin{+\!\!+} \vec{m}_2'$
2. $\forall b : B.\ [\langle e_{\mathcal{A}}, b\rangle] \approx [\,]$
3. $\forall a_1, a_2 : A, b : B.\ [\langle a_1, b\rangle, \langle a_2, b\rangle] \approx [\langle a_1 * a_2, b\rangle]$
4. $\forall a : A.\ [\langle a, e_{\mathcal{B}}\rangle] \approx [\,]$
5. $\forall a : A, b_1, b_2 : B.\ [\langle a, b_1\rangle, \langle a, b_2\rangle] \approx [\langle a, b_1 * b_2\rangle]$

We use requirement 1 to impose a monoidal structure $\mathcal{A} \otimes \mathcal{B}$ on the quotient set $\frac{\mathbb{L}(A \times B)}{\approx}$:

Operator $\frac{+\!\!+}{\approx} = \lambda q_1, q_2.\ \texttt{select}\ \vec{m}_1\ \texttt{from}\ q_1\ \texttt{in}\ (\texttt{select}\ \vec{m}_2\ \texttt{from}\ q_2\ \texttt{in}\ \frac{\vec{m}_1 +\!\!+ \vec{m}_2}{\approx}\ \texttt{using}\ \textbf{.})\ \texttt{using}\ \textbf{.}$

Associativity Follows from associativity of $+\!\!+$ and the fact that quotienting only makes things more equal

Identity Element $= \frac{[\,]}{\approx}$

Identity Follows from identity of $[\,]$ and the fact that quotienting only makes things more equal

Exercise 4. Show that, for any monoid $\mathcal{C}$, there is a bijection between the set of multilinear homomorphisms from $\mathcal{A}$ and $\mathcal{B}$ to $\mathcal{C}$ and the set of monoid homomorphisms from $\mathcal{A} \otimes \mathcal{B}$ to $\mathcal{C}$.

Notation. $\mathcal{A} \otimes \mathcal{B}$ is called the *tensor (product)* of $\mathcal{A}$ and $\mathcal{B}$.