Adjunctions

Ross Tate

September 29, 2014

Exercise 1. Prove that the inclusion functor $\mathbf{Set} \overset{I}{\hookrightarrow} \mathbf{Rel}$ has a right adjoint. You may use any of the equivalent definitions of adjunction. For clarification, I is the functor mapping each set X (an object of **Set**) to the set X (also an object of **Rel**) and each function $X \to Y$ (a morphism of **Set**) to the relation $\lambda\langle x, y\rangle.\, f(x) = y$ (a morphism of **Rel**).

Exercise 2. There is a functor from **1** to **Set** picking out the empty set, and another functor from **1** to **Set** picking out the singleton set. One is the left adjoint to the unique functor from **Set** to **1**, and the other is the right adjoint to the unique functor from **Set** to **1**. Determine and prove which is which.

Exercise 3. $\mathbb{N} : \mathbf{1} \to \mathbf{Set}$ maps the only object of **1** to the set $\mathbb{N}$. `repeat` is the natural transformation from $\mathbb{N}$ to $\mathbb{N}\,;\mathbb{L}$ (i.e. $\mathbb{L}(\mathbb{N})$) mapping the sole object of **1** to the function mapping n to the length-n list $[n, \ldots, n]$. `sum` is the natural transformation from $\mathbb{N}\,;\mathbb{L}$ to $\mathbb{N}$ mapping the sole object of **1** to the function mapping a list of numbers and returns its sum.

The string diagram to the right denotes a natural transformation from the functor $\mathbb{N} : \mathbf{1} \to \mathbf{Set}$ to itself ($\mathbb{N}$ maps the only object of **1** to the set $\mathbb{N}$). In particular, this means it describes a function from $\mathbb{N}$ to $\mathbb{N}$. Determine what that function is in terms of basic arithmetic. (No proof necessary; the purpose of this is to learn the notation.)