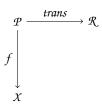
## Proofs

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**Exercise 1.** Let *trans* :  $\langle \{a, b, c\}, \{\langle a, b \rangle, \langle b, c \rangle \} \rangle \rightarrow \langle \{a, b, c\}, \{\langle a, b \rangle, \langle b, c \rangle, \langle a, c \rangle \} \rangle$  be a morphism of **Rel**(2) whose underlying function is the identity. Call its domain  $\mathcal{P}$  and its codomain  $\mathcal{R}$ . Describe, in standard set-theoretic terms, what the pushout of the following is, for any object  $\mathcal{X}$  and morphism  $f : \mathcal{P} \rightarrow \mathcal{X}$ :



Prove that your description is actually a pushout.

*Proof.* Let  $\mathcal{Y} = \langle X, R_{\chi} \cup \{ \langle f(a), f(c) \rangle \} \rangle$ . Let  $\kappa_{\chi}$  be the identity function, which is obviously relation preserving. Let  $\kappa_{\mathfrak{K}}$  be the underlying function of f, which is relation preserving because f is relation preserving and f(a) is related to f(c) in  $\mathcal{Y}$  by definition.

Suppose there is an object Z with morphisms  $g: X \to Z$  and  $h: \mathcal{R} \to Z$  such that f; g equals *trans*; h. Then, for  $\kappa_X; [g, h]$  to equal g, the underlying function of [g, h] must be the underlying function of g because the underlying function of  $\kappa_X$  is the identity, which guarantees uniqueness of [g, h]. This also implies that  $\kappa_{\mathcal{R}}; [g, h]$  equals h, since the underlying function of  $\kappa_{\mathcal{R}}$  is f, and f; g equaling *trans*; h implies f; g equals h. For existence, we must prove that g is relation-preserving from  $\mathcal{Y}$  to Z and that  $\kappa_{\mathcal{R}}; g$  equals h. Given two elements related in  $\mathcal{Y}$ , by the definition of the relation of  $\mathcal{Y}$ , they must either be related in X or they must be the pair  $\langle f(a), f(c) \rangle$ . The former case is preserved because g is relation-preserving from X to Z. For the latter case, g(f(a)) equals h(a) and g(f(c)) equals h(c), and aand c are related in  $\mathcal{R}$ , so h being relation-preserving implies that g(f(a)) must be related to g(f(c)) in Z.