STUDIES IN LOGIC
AND
THE FOUNDATIONS OF MATHEMATICS

L. E. J. BROUWER
E. W. BETH
A. HEYTING

Editors

These three Dutch editors were the leaders of the intuitionistic school of mathematics.

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RECURSIVE NUMBER THEORY

A DEVELOPMENT OF RECURSIVE ARITHMETIC IN A LOGIC-FREE EQUATION CALCULUS

BY

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1912 – 1985

He expressed doubts about both the formalist and intuitionist (finitist) approaches to mathematics, and this led him to take up an extreme finitist position which, roughly speaking, consisted of those parts of mathematics that are accepted in all schools of thought. He was greatly influenced by Wittgenstein whose lectures he attended in Cambridge between 1931 and 1935.

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PREFACE

The discovery of the reflexive paradox, that the class of all classes which are not members of themselves is a self-contradictory concept, gave rise to three new developments in mathematics. The first of these was Russell's theory of types, one part of which segregated objects into types (classes of objects of one type forming the objects of the next higher type) and prohibited the formation of classes of mixed type. This theory led to considerable complications in the construction of arithmetic since it excluded not only paradoxes but also constructions fundamental to the theory of real numbers, like that of the least upper bound of a bounded class of numbers, and the rehabilitation of these constructions necessitated the introduction of an axiom which associated with a propositional function (propositional form with a variable), whose argument ranged over objects of a given type, a propositional function with the same truth values whose argument ranged over objects of the first type. In more recent formulations of set theory alternatives to type theory (and the reducibility axiom) have been proposed; these alternative treatments depend upon some restriction of the right of membership, fortified (in the case of Quine's system) by what may be called a potential typing, according to which the object symbols in every valid formula in a type-free symbolism must admit an assignment of numbers in such a way that each object receives a number which is one less than the number of the class to which it belongs.

The second development which was initiated by the discovery of the reflexive paradox was Brouwer's 'intuitionistic' logic and arithmetic, the most novel feature of which was the denial of the *tertium-non-datur*, the principle of logic which asserts that every proposition is either true, or false, no third possibility presenting itself. The rejection of the *tertium-non-datur* eliminates the reflexive paradox since the paradox rests on the assumption that every class

either is, or is not, a member of itself, but it also invalidates the familiar *interpretations* of a considerable part of arithmetic (although Gödel has shown that intuitionistic arithmetic includes the whole of classical arithmetic, in the sense that to any formula provable by classical logic corresponds a formula provable by intuitionistic logic).

The third system which was developed to escape the reflexive paradox was Skolem's *recursive* arithmetic. Skolem observed that he could avoid the paradox without recourse to the restrictions of type theory and without the rejection of any rule of classical logic if he did not take *existence* as one of the primitive notions of logic. In a calculus which expresses universality only by means of free variables this has the effect of *preventing* the application of the *tertium non datur* in those cases where it might lead to paradox, since it rules out the negation of universal propositions.

The sacrifice of existence as a primitive notion deprived Skolem of the classical method of function definition and in its place he introduced *definition by recursion*. A function $f(n)$ is said to be defined by recursion if, instead of being defined explicitly, (that is, as an abbreviation for some other expression), only the value of $f(0)$ is given, and $f(n+1)$ is expressed as a function of $f(n)$. In other words a recursive definition does not define $f(n)$ itself, but provides a process whereby the values of $f(0)$, $f(1)$, $f(2)$, $f(3)$ and so on, are determined one after the other.

In the following account of recursive arithmetic we show that logic and arithmetic may be constructed simultaneously in a free variable equation calculus in which the only statements are equations of the form $a=b$, where "a" and "b" stand for function signs. By means of this equation calculus the *ab initio* construction of logic and arithmetic *may be presented in full rigour and detail* at a far more elementary level than has hitherto been possible, and it is hoped that the first half of the book will prove to be suitable for the mathematical specialist in his first year at the University. In this part a great deal of the smaller detail has been separated from the text and presented in example form (with complete solutions at the end of the book) both to make the text lighter to read and to help the reader to acquire a new technique in easy stages.

On page 3 he argues that numbers are clearly not the same as numerals. (See Martin-Löf as well.)

numbers may be even or odd, prime or composite but these are not properties of numerals. A more sophisticated version of this attempt to define numbers in terms of numerals, makes numbers, not the same thing as, but the *names* of the numerals; this escapes the absurdities which arise in attempting to *identify* number and numeral but it leads to the equally absurd conclusion that some one *notation* is the quintessence of number. For if numbers are the names of numerals then we must decide which numerals they name; we cannot accept the number ten for instance as both the name of the roman numeral and the arabic numeral. And if it is said that the number ten is the name of all the numerals ten then we reach the absurd conclusion that the meaning of a number word changes with each notational innovation.

The antithesis of "number" and "numeral" is one which is common in language, and perhaps its most familiar instance is to be found in the pair of terms "proposition" and "sentence". The sentence is some physical representation of the proposition, but cannot be identified with the proposition since different sentences (in different languages, for instance) may express the same proposition. If, however, we attempt to say just what it is that the sentences express we find that the concept of proposition is just as difficult to characterise as the concept of number. It is sometimes held that the proposition is something in our minds, by contrast with the sentence, which belongs to the external world, but if this means that a proposition is some sort of mental image then it is just another instance of the confusion of a proposition with a sentence, for whatever may be in our minds, whether it be a thought in words, or a picture, or even some more or less amorphous sensation, is a *representation* of the proposition, differing from the written or spoken word only because it is not a communication. In the same way we see that the view that number is indefinable, being something which we know by our intuition, again confuses number with numeral, that is confuses number with one of its representations.

Arithmetic and the Game of Chess

The game of chess, as has often been observed, affords an excellent parallel with mathematics (or, for that matter, with language itself). To the numerals correspond the chess pieces, and to the operations of arithmetic, the moves of the game. But the parallel is even closer than this, for to the problem of defining number corresponds the problem of defining the entities of the game. If we ask ourselves the question "What is the king of chess?" we find precisely the same difficulties arise in trying to find an answer which we met in our consideration of the problem of defining the concept of number. Certainly the king of chess, whose moves the rules of the game prescribe, is not the piece of characteristic shape which we *call* the king, just as a numeral is not a number, since any other object, a matchstick or a piece of coal, would serve as well to play the king in any game. Instead of the question "What is the king of chess?" let us ask "What makes a particular piece in the game the king piece?" Clearly it is not the shape of the piece or its size, since either of these can be changed at will. What constitute a piece king are its *moves*. That piece is king which has the king's moves. And the king of chess itself? The king of chess is simply one of the *parts* which the pieces play in a game of chess, just as King Lear is a part in a drama of Shakespeare's; the actor who plays the King is King in virtue of the *part* which he takes, the sentences he speaks and the actions he makes, (and not simply because he is dressed as king) and the piece on the chess board which plays the king-role in the game is the piece which makes the king's moves.

Here at last we find the answer to the problem of the nature of numbers. We see, first, that for an understanding of the meaning of numbers we must look to the 'game' which numbers play, that is to arithmetic. The numbers, one, two, three, and so on, are characters in the game of arithmetic, the pieces which play these characters are the numerals and what makes a sign the numeral of a particular number is the part which it plays, or as we may say in a form of words more suitable to the context, what constitute a sign the sign of a particular number are the *transformation rules* of the sign. It follows, therefore, that the OBJECT OF OUR STUDY IS NOT NUMBER ITSELF BUT THE TRANSFORMATION RULES OF THE NUMBER SIGNS, and in the chapters which follow we shall have no further occasion to refer to the number concept. But just as the rules of chess are currently formulated in terms of the entities of chess, so that we say, for instance, the king of chess moves only one square at a time (except in castling), instead of the completely equivalent formulation "the piece playing the part of king (or simply the king-piece) is moved only one square at a time (except in castling)" so we shall continue, in purely descriptive passages, to formulate the operations of arithmetic in terms of arithmetical entities instead of arithmetical signs. For instance, we may speak of "the sum of the numbers two and three" rather than confine ourselves to the object formulation "$2+3$", where $+$ is the sign whose *role* in arithmetic is what is called addition, and "2" and "3" are numerals whose roles are those of the numbers two and three. To put it another way, in defining the part played by a sign like $+$, in arithmetic, we shall say that what we are defining is the sum function, but the definition itself will refer only to operations for transforming *expressions* which contain the sign $+$.

Number Variables

The parallel between chess and arithmetic breaks down when we contrast the predetermined set of pieces in the game of chess with the licence granted to arithmetic to construct numerals at will. In this respect arithmetic more closely resembles a *language* which places no limit, in principle, upon the length of its words. A familiar notation for numerals expresses them as words spelt with the 'alphabet' "0", "1" and "$+$"; each 'word' has an initial "0" followed by a succession of pairs "$+1$". Thus, for instance, we form in turn "0", "$0+1$", "$0+1+1$", "$0+1+1+1$". The formation of numerals may be *fully* characterised by means of two operations, as follows. We extend the alphabet by the introduction of a new sign, "x", and form 'words' by writing either "0" or "$x+1$" for "x"; for example we may form in turn, "x", "$x+1$", "$x+1+1$", "$x+1+1+1$", "$0+1+1+1$", the last of which is a numeral. This new sign we call a 'numeral variable'. The rules permitting the substitution of "$x+1$" or "0" for "x" in effect allow the substitution of *any numeral* for x; the object of the formulation we have adopted is that it serves to define the concept of *any numeral* and the concept of a *numeral variable* simultaneously. In the sequel, not only the letter x, but other letters, too, will be used as numeral variables.

The numeral formed by writing some numeral for "x" in "$x+1$" is called the *successor* of that numeral. For instance, writing "$0+1+1$" for "x" in "$x+1$" we obtain "$0+1+1+1$", the successor of "$0+1+1$". For this reason "$x+1$" is called the (sign of the) *successor function*. The definite article is somewhat misleading, however, since we may write, in place of x, any other letter which is being used as a numeral variable; in a system in which x, y and z are all numeral variables, each of "$x+1$", "$y+1$", "$z+1$" is a sign of the successor function. Nevertheless we shall talk of *the* successor function, the uniqueness in question being the uniqueness of the sign which results when we write some definite numeral for the variable, be it denoted by x, y or z.

For purposes of standardisation of notation we shall have occasion to introduce, instead of the 'alphabet' "0", "1" and "$+$" for writing numerals, the 'alphabet' "0", "S" in which the numerals become "0", "$S0$", "$SS0$", "$SSS0$" and so on. In this notation the sign of the successor function is "Sx" and the transformation rules for a numeral variable x are (i) Sx may be written for x, (ii) 0 may be written for x.

Another notation in current use employs "x'" for the successor function, so that the numerals are written "0", "$0'$", "$0''$", "$0'''$" and so on.

Definition of Counting

No theory of the natural numbers is complete which does not also take into account the part which numbers play *outside* arithmetic. It is not only a property of the number nine that it is a square but also that it is the number of the planets, and this

We have already seen that Per Martin-Löf bases his account of the type N of natural numbers on ideas similar to those advanced by Thoralf Skolem in 1923 and explained here by Goodstein (Reuben Louis Goodstein).

Next Goodstein develops a very simple proof system for establishing equalities between natural numbers. It is easy to imagine building such a proof system for the programming language of primitive recursive functions because we know what equality means on numbers.

Most programming languages are not explicit about equality, but that is a small step in some cases to converting the language into a logical system as we see here. This step is harder for languages with a rich type system, and we will gradually undertake it.

CHAPTER II

THE EQUATION CALCULUS

2. We have described the two processes, substitution and recursion, by which new functions may be defined from functions previously introduced, and have shown how all recursive functions are built up from the zero and successor functions. We turn now from the construction of functions to the *proofs* of equations. The equations we consider express the fact that certain function signs may be substituted for others; for instance the equation $x+y=y+x$ says that $x+y$ may be substituted for $y+x$, and *vice-versa*. The general form of such equations is $F=G$ where F and G are recursive functions.

2.1 Definition of a Proof

A *proof* is a table of equations each of which is either (part of) the definition of a function, or an equation of the form $F=F$, or is a *proved* equation. If $F=G$ is one of the equations of a proof, then a proved equation is obtained by replacing the function F by the function G at one or more of the places at which F occurs in some equation of a proof.

Furthermore, the equation formed by replacing a variable at all the points at which it occurs in some equation of a proof, by another variable, or by a definite numeral or function, is a proved equation.

Finally, $F=G$ is a proved equation if equations of a proof are obtained by substituting the function F for a function H, and by substituting the function G for H, in the equations which define the function H.

If the results of substituting F and G in the defining equations of some function are equations of a proof, we shall say that F and G *satisfy the same introductory equations*. The rule that $F=G$

if F and G satisfy the same introductory equations is just another way of saying that the processes of recursion and substitution define *unique* functions. For instance, the equations $x+0=x$, $x+Sy=S(x+y)$, which introduce the sum function, define this function completely, so that any $f(x, y)$ which satisfies the same equations, namely $f(x, 0)=x$, $f(x, Sy)=Sf(x, y)$, is just another notation for the same function.

2.11 To illustrate the proof technique, and in preparation for subsequent applications, we collect together here proofs of the principal properties of the functions we introduced in the previous chapter.

Properties of the Sum Function

As the first example of a proof we consider the commutative property of addition which is expressed by the equation

2.2 $$x+y=y+x$$

To make it easier to follow we divide the proof into two parts, starting with the proof of the equation

$$Sx+y=S(x+y)$$

The proof consists of the nine equations

(2.21) $$x+0=x$$

(2.22) $$Sx+0=Sx$$

(2.23) $$Sx=Sx$$

(2.24) $$S(x+0)=Sx$$

(2.25) $$x+Sy=S(x+y)$$

(2.26) $$Sx+Sy=S(Sx+y)$$

(2.27) $$S(x+Sy)=S(x+Sy)$$

(2.28) $$S(x+Sy)=SS(x+y)$$

(2.29) $$Sx+y=S(x+y).$$

We observe that (2.21), (2.25) are defining equations of the sum function; (2.22) is derived from (2.21) by substituting Sx for x; (2.24) derives from (2.23) by substituting $x+0$ for x (on the left hand side) as is permitted by (2.21); (2.26) derives from (2.25) by substituting Sx for x; (2.28) is obtained from (2.25) and (2.27). The equations (2.22), (2.24), (2.26) and (2.28) show that both $Sx+y$ and $S(x+y)$ satisfy the introductory equations

$$\varphi(x, 0)=Sx$$
$$\varphi(x, Sy)=S\varphi(x, y)$$

so that (2.29) is a proved equation. Substituting x for y and y for x in (2.29) we obtain

2.291 $$Sy+x=S(y+x)$$

The next step is the proof of the equation

2.3 $$0+x=x\;;$$

the pair of equations

$$0+0=0 \qquad 0+Sx=S(0+x)$$

which follow by substitution in (2.21) and (2.25), show that the function $0+x$ satisfies the introductory equations

$$\varphi(0)=0 \qquad \varphi(Sx)=S\varphi(x)$$

which are also satisfied by the identity function $\mathscr{I}(x)$, proving 2.3. The four equations 2.21, 2.3, 2.25 and 2.291 show that $x+y$ and $y+x$ both satisfy the introductory equations

$$\varphi(x, 0)=x \qquad \varphi(x, Sy)=S\varphi(x, y)$$

which completes the proof that

$$x+y=y+x.$$

We take as the next example the important equation

2.4 $$(x+y)+z=x+(y+z)$$

which expresses the associative property of addition.

Since $(x+y)+Sz=S\{(x+y)+z\}$ and

$$x+(y+Sz)=x+S(y+z)=S\{x+(y+z)\}$$

(where we are using an abbreviation of the form $a=b=c$ to denote the pair of equations $a=b$, $b=c$)

it follows that both the functions $(x+y)+z$ and $x+(y+z)$ satisfy the equation $\varphi(x, y, Sz)=S\varphi(x, y, z)$;

furthermore $(x+y)+0=x+y=x+(y+0)$

so that these functions also satisfy the equation

$$\varphi(x, y, 0)=x+y$$

and so 2.4 is proved.

2.5 Products

The commutative property of multiplication, which is expressed by the equation

$$xy=yx$$

is readily proved with the help of equations 2.2 and 2.4. The defining equations of the product yx are

2.51 $$0\cdot x=0\ ,\ (Sy)\cdot x=(y\cdot x)+x\ ;$$

as in the proof of 2.2 we start by establishing the companion equations

2.52 $$x\cdot 0=0\ ,\ x\cdot Sy=(x\cdot y)+x\ .$$

The first of these is proved by the two equations $0\cdot 0=0, Sx\cdot 0=x\cdot 0$ (instances of the introductory equations of the product), which in conjunction with $Z(0)=0, ZSx=Zx$ (where Zx is the zero function), show that

$$x\cdot 0=Zx=0\ .$$

For the second we observe that

$$0\cdot Sy=0=0\cdot y+0\ ,\quad Sx\cdot Sy=x\cdot Sy+Sy$$

and
$$\begin{aligned} Sx\cdot y+Sx &= (x\cdot y+y)+Sx \\ &= (x\cdot y+Sx)+y && \text{by } 2.4 \\ &= S(x\cdot y+x)+y \\ &= S((x\cdot y+x)+y) && \text{by } 2.29 \\ &= (x\cdot y+x)+Sy \end{aligned}$$

which shows that $x\cdot Sy$ and $x\cdot y+x$ both satisfy the introductory equations

$$\varphi(0, y)=0 \qquad \varphi(Sx, y)=\varphi(x, y)+Sy,$$

proving 2.52. The equations 2.51, 2.52 show that the functions $y\cdot x$ and $x\cdot y$ satisfy the same introductory equations, which completes the proof.

2.53 If for some $f(x)$, $f(0)=0$ and $f(Sx)=0$ are provable then $f(x)=0$ is provable. For from $f(Sx)=0$ and $0\cdot f(x)=0$ we derive $f(Sx)=0\cdot f(x)$ and since $ZSx=0\cdot Zx$ therefore $f(x)$ and Zx both satisfy the introductory equations

$$\varphi(0)=0\ ,\ \varphi(Sx)=0\cdot\varphi(x)$$

so that $f(x)=Zx$ and $f(x)=0$ is provable.

2.6 Properties of the Difference Function

We commence by proving the important equation

2.61 $$x\dot{-}y=Sx\dot{-}Sy.$$

From the equations

$$x\dot{-}0=x\ ,\ Sx\dot{-}S0=(Sx\dot{-}0)\dot{-}1=Sx\dot{-}1=x\ ,$$

and $$x\dot{-}Sy=(x\dot{-}y)\dot{-}1\ ,\ Sx\dot{-}SSy=(Sx\dot{-}Sy)\dot{-}1$$

(which are instances of the introductory equations of the difference function) it follows that $x\dot{-}y$ and $Sx\dot{-}Sy$ both satisfy the introductory equations

$$\varphi(x, 0)=x\ ,\ \varphi(x, Sy)=\varphi(x, y)\dot{-}1$$

so that $$x\dot{-}y=Sx\dot{-}Sy\ .$$

It follows that

2.611 $$(x+y)\dot{-}y=x$$

for $(x+Sy)\dot{-}Sy=S(y+x)\dot{-}Sy=(x+y)\dot{-}y$ and therefore if $I_1(x, y)$ is defined explicitly by the equation $I_1(x, y)=x$ both $(x+y)\dot{-}y$ and $I_1(x, y)$ satisfy

$$\varphi(x, 0)=x\ ,\ \varphi(x, Sy)=\varphi(x, y).$$

From the equations $0\dot{-}0=0$, $0\dot{-}Sx=(0\dot{-}x)\dot{-}1$ and $Z0=0$, $ZSx=Zx\dot{-}1$ follows $0\dot{-}x=0$ and similarly using $Z(x, y)$ defined explicitly by the equation $Z(x, y)=0$ we may prove $y\dot{-}(x+y)=0$, and thence $y\dot{-}y=0$.
We *define* the positive difference of x and y $|x, y|$ by the equation

$$|x, y|=(x\dot{-}y)+(y\dot{-}x)\ .$$

It follows that $|x, y|=|y, x|$, and using 2.61, that

2.612 $$|x, y|=|Sx, Sy|\ .$$

The only applications of the equalising rules for functions which we have made so far have been confined to single recursions. However, the following proof of the equation

2.62 $$x+(y\dot{-}x)=y+(x\dot{-}y)$$

requires an application of the rule to a double recursion. We observe that

$$x+(0\dot{-}x)=x=0+(x\dot{-}0)$$

$$0+(Sy\dot{-}0)=Sy=Sy+(0\dot{-}Sy)$$

and $$Sx+(Sy\dot{-}Sx)=Sx+(y\dot{-}x)=S(x+(y\dot{-}x))$$

$$Sy+(Sx\dot{-}Sy)=Sy+(x\dot{-}y)=S(y+(x\dot{-}y))$$

so that $x+(y\dot{-}x)$ and $y+(x\dot{-}y)$ both satisfy the doubly recursive equations

$$\varphi(x, 0)=x\ ,\ \varphi(0, Sy)=Sy$$

$$\varphi(Sx, Sy)=S\varphi(x, y)$$

which completes the proof. The common value of $x+(y\dot{-}x)$ and $y+(x\dot{-}y)$ is the greater of x and y, denoted by max (x, y).

In preparation for a theorem connecting equality with zero positive difference we prove next that for any recursive function $f(x)$

2.63 $$(1\dot{-}|x, y|)f(x)=(1\dot{-}|x, y|)f(y)\ .$$

The special case of 2.63 obtained by substituting $x+y$ for x, namely

2.631 $$(1\dot{-}x)f(x+y)=(1\dot{-}x)f(y)$$

is evident, since $(1\dot{-}x)f(x+y)$ and $(1\dot{-}x)f(y)$ both satisfy

$$\varphi(0, y)=f(y)\ ,\ \varphi(Sx, y)=Z\varphi(x, y)\ ;$$

substituting $x\dot{-}y$ for x in 2.631 we find

2.64 $$(1\dot{-}(x\dot{-}y))f(y)=(1\dot{-}(x\dot{-}y))f(y+(x\dot{-}y))$$

and after multiplying by $1\dot{-}|x, y|$ and using the provable equation $\{1\dot{-}(x\dot{-}y)\}\{1\dot{-}|x, y|\}=\{1\dot{-}|x, y|\}$, (see Example 2.241), this becomes

2.65 $$(1\dot{-}|x, y|)f(y)=(1\dot{-}|x, y|)f(y+(x\dot{-}y))\ .$$

Substituting x for y and y for x in 2.65, since $|x, y|=|y, x|$, we obtain

2.66 $$(1\dot{-}|x, y|)f(x)=(1\dot{-}|x, y|)f(x+(y\dot{-}x))$$

and so 2.63 follows by 2.62.

Taking $f(x)$ to be the identity function $\mathscr{I}(x)=x$ in 2.63 we have

2.67 $$(1\dot{-}|x, y|)x=(1\dot{-}|x, y|)y$$

and so if $|F, G|=0$ is a proved equation for certain functions F, G then substituting F, G for x, y in 2.67 we see that $F=G$ is a proved equation. Conversely from $F=G$ we derive, by means of $F\dot{-}F=0$, both $F\dot{-}G=0$ and $G\dot{-}F=0$ and so $|F, G|=0$.

Thus we have shown that each of the equations

$$|F, G|=0\ ,\ F=G$$

may be deduced from the other.

The equation 2.63 may be expressed in another form which is often more convenient to use; if we multiply 2.63 by $1 \dot{-} f(x)$, and use the equation

$$(1 \dot{-} f(x))f(x) = 0$$

which is obtained by substituting $f(x)$ for x in the provable equation $x(1 \dot{-} x) = 0$ (see Example 2.1) we obtain

2.68 $$(1 \dot{-} |x, y|)(1 \dot{-} f(x))f(y) = 0.\,*$$

2.7 As a first example of the use of the equivalence of the equations $|F, G| = 0$ and $F = G$ we prove the equality of two functions f and g which are known to satisfy the equations

$$f(0) = g(0)\ ,\ f(Sx) = g(Sx)\ .$$

We define $\varphi(x) = |f(x)\ ,\ g(x)|$ so that $\varphi(0) = 0\ ,\ \varphi(Sx) = 0$ are proved equations, and therefore $\varphi(Sx) = Z\varphi(x)$. Since Zx also satisfies the introductory equations

$$Z0 = 0\ ,\ ZSx = ZZx$$

therefore $\varphi(x) = Zx$, and so $\varphi(x) = 0$, are proved equations, whence, by the previous theorem, $f(x) = g(x)$ is a proved equation. A generalisation of this result is given in example 2.73.

To record the fact that a proof of some equation $F = G$ may be obtained by combining the proofs of one or more equations $f_1 = g_1,\ f_2 = g_2,\ \ldots\ ,\ f_k = g_k$, we write simply

$$\begin{array}{c} f_1 = g_1 \\ f_2 = g_2 \\ \cdots\cdots\cdot \\ f_k = g_k \\ \hline F = G \end{array}$$

and call this a *proof schema*.

* Multiplying an equation by a factor is a shorthand way of describing the derivation of an equation $ca = cb$ from the equation $a = b$; this derivation is of course effected by substituting "b" for "a" on the right-hand side of the equation $ca = ca$.

Thus the foregoing theorem may be represented by the schema

$$\begin{array}{c} f(0) = g(0) \\ f(Sx) = g(Sx) \\ \hline f(x) = g(x)\ . \end{array}$$

2.8 If for some function $f(x)$ the equations

$$f(0) = 0\ ,\ (1 \dot{-} f(x))f(Sx) = 0$$

are provable, then the equation $f(x) = 0$ is provable; in schematic form

$$\begin{array}{c} f(0) = 0 \\ (1 \dot{-} f(x))f(Sx) = 0 \\ \hline f(x) = 0\ . \end{array}$$

By example 2.232, $(1 \dot{-} a)b = b \dot{-} ab$ and so from $(1 \dot{-} f(x))f(Sx) = 0$ we deduce

$$(1 \dot{-} f(x))(1 \dot{-} f(Sx)) = (1 \dot{-} f(x)) \dot{-} (1 \dot{-} f(x))f(Sx) = 1 \dot{-} f(x),$$

and so if $g(x)$ is defined by the recursion

$$g(0) = 1\ ,\ g(Sx) = g(x)(1 \dot{-} f(x))$$

and if $h(x) = g(Sx)$ then

$$h(Sx) = g(Sx)(1 \dot{-} f(Sx)) = g(x)(1 \dot{-} f(x)) = g(Sx)\ ;$$

but $h(0) = g(0)$ (since $f(0) = 0$) and so $h(x) = g(x)$;
Thus $g(0) = 1$, $g(Sx) = g(x)$ and so (by example 2.7304), $g(x) = 1$, and therefore $1 \dot{-} f(x) = 1$.
Hence $f(x) = f(x)(1 \dot{-} f(x)) = 0$, using example 2.1.
We shall show later that theorem 2.8 establishes the validity of the method of proof known as mathematical induction.

2.9 The Functions Σ_f, Π_f and μ_f.

For any function $f(x)$ we define

$$\Sigma_f(0) = f(0)\ ,\ \Sigma_f(n+1) = \Sigma_f(n) + f(n+1)$$
$$\Pi_f(0) = f(0)\ ,\ \Pi_f(n+1) = \Pi_f(n) \cdot f(n+1)$$

so that for any numeral p

$$\Sigma_f(\mathrm{p}) = f(0) + f(1) + \ldots + f(\mathrm{p})$$

and $$\Pi_f(\mathrm{p}) = f(0) \cdot f(1) \cdot \ldots \cdot f(\mathrm{p}) ;$$

for a function $f(x, a)$ depending also upon a parameter a we write

$$\Sigma_f(0, a) = f(0, a)$$

$$\Sigma_f(n+1, a) = \Sigma_f(n, a) + f(n+1, a)$$

with an analogous formulation for the case of more than one parameter. We take for granted the companion definitions for the function Π.
For any assigned numeral p we have

$$\Sigma_f(\mathrm{p}, a) = f(0, a) + f(1, a) + \ldots + f(\mathrm{p}, a)$$

and $$\Pi_f(\mathrm{p}, a) = f(0, a) \cdot f(1, a) \cdot \ldots \cdot f(\mathrm{p}, a) .$$

In using the functions Σ_f, Π_f we take the *first* variable in f to be the operative variable; to study such a sum as

$$f(a, 0) + f(a, 1) + \ldots + f(a, \mathrm{p})$$

we introduce $\varphi(x, a)$ by the explicit definition $\varphi(x, a) = f(a, x)$ and express $f(a, 0) + f(a, 1) + \ldots + f(a, \mathrm{p})$ as $\Sigma_\varphi(\mathrm{p}, a)$.

2.91 If for some functions $f(x)$, $g(x)$ the equation

$$(1 \dot{-} g(n))(Sn \dot{-} a)f(a) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(i)}$$

is provable, then

$$(1 \dot{-} g(n))\Sigma_f(n) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(ii)}$$

is provable.

Denote $(Sn \dot{-} m)(1 \dot{-} g(n))\Sigma_f(m)$ by $A(m)$; then by (i) $A(0) = 0$, and

$$A(Sm) = (Sn \dot{-} Sm)(1 \dot{-} g(n))\{\Sigma_f(m) + f(Sm)\}$$

$$= (Sn \dot{-} Sm)(1 \dot{-} g(n))\Sigma_f(m)$$

$$= A(m) \dot{-} (1 \dot{-} g(n))\Sigma_f(m)$$

so that $\{1 \dot{-} A(m)\}\, A(Sm) = 0$, and therefore by 2.8, $A(m) = 0$.
In particular $A(n) = 0$, that is, $\{1 \dot{-} g(n)\}\Sigma_f(n) = 0$.

2.92 If for some $f(x)$, $g(x)$

$$(Sn \dot{-} a)(1 \dot{-} f(a))\, g(n) = 0$$

then $\{1 \dot{-} \Pi_f(n)\}\, g(n) = 0$.
Let $A(m)$, $B(m)$, $C(m)$ denote respectively

$$(Sn \dot{-} m)(1 \dot{-} \Pi_f(m))\, g(n) \; , \; (Sn \dot{-} Sm)(1 \dot{-} \Pi_f(m))\, g(n)$$

and $(Sn \dot{-} Sm)(1 \dot{-} f(Sm))\, g(n)$.
By hypothesis $C(m) = 0$ and so $\{1 \dot{-} A(m)\}\, C(m) = 0$; moreover

$$\{1 \dot{-} A(m)\}\, B(m) = \{1 \dot{-} A(m)\}\, \{A(m) \dot{-} (1 \dot{-} \Pi_f(m))\, g(n)\} = 0$$

and so (by example 2.47, with $\Pi_f(m)$ for a, $f(Sm)$ for b and $(Sn \dot{-} Sm)g(n)$ for c),

$$\{1 \dot{-} A(m)\}\, A(Sm) = 0,$$

and therefore $A(m) = 0$, whence $A(n) = 0$, that is

$$\{1 \dot{-} \Pi_f(n)\}\, g(n) = 0 .$$

2.921 $$\{1 \dot{-} f(x)\}\, \Pi_f(x) = 0 .$$

For $$\{1 \dot{-} f(0)\}\, \Pi_f(0) = \{1 \dot{-} f(0)\}\, f(0) = 0$$

and $$\{1 \dot{-} f(Sx)\}\, \Pi_f(Sx) = \{1 \dot{-} f(Sx)\}\, f(Sx) \cdot \Pi_f(x) = 0.$$

2.93 In the following section we require a number of the elementary properties of the function $\alpha(x) = 1 \dot{-} (1 \dot{-} x)$ which are proved in *Examples* 2.26. In particular we shall use the equations

$$\alpha(\alpha(x)) = \alpha(x) \; , \; \alpha(1 \dot{-} x) = 1 \dot{-} \alpha(x) \; , \; x \cdot \alpha(x) = x$$

and $$\alpha(xy) = \alpha(x) \cdot \alpha(y) .$$

The function $\alpha(x)$ obviously satisfies the equations

$$\alpha(0) = 0 \; , \; \alpha(Sx) = 1 .$$

For some given $f(x)$ we define $\varrho(x) = \alpha(f(x))$. It follows that

$$\alpha(\varrho(x)) = \alpha(\alpha(f(x))) = \alpha(f(x)) = \varrho(x)$$

and $$\alpha(1 \dot{-} \varrho(x)) = 1 \dot{-} \alpha(\varrho(x)) = 1 \dot{-} \varrho(x) .$$

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