1 Bi-simulation and congruency in π-calculus

$$
\begin{array}{rcl}
\pi & ::= & \bar{x}\,y \mid x(z) \mid \tau \mid [x = y]\pi \\
P & ::= & M \mid P_1|P_2 \mid \nu x P \mid !P \\
M & ::= & 0 \mid \pi.P \mid M_1 + M_2
\end{array}
$$

We write labeled transitions for processes as $P \xrightarrow{\alpha} P'$ where α is taken from

$$\alpha ::= \bar{x}\,y \mid x(z) \mid \bar{x}(z) \mid \tau$$

$\bar{x}(z)$ is a "bound send" and is used to identify when scope extrusion occurs.

1.1 LTS Operational semantics

$$\bar{x}\,y.P \xrightarrow{\bar{x}y} P \qquad \text{(Out)}$$

$$x(z).P \xrightarrow{x(y)} P[{}^{y}/{}_{z}] \qquad \text{(InP)}$$

$$\tau.P \xrightarrow{\tau} P \qquad \text{(Tau)}$$

$$\frac{P \xrightarrow{\alpha} P'}{P + Q \xrightarrow{\alpha} P'} \qquad \text{(Sum-L)}$$

$$\frac{Q \xrightarrow{\alpha} Q'}{P + Q \xrightarrow{\alpha} Q'} \qquad \text{(Sum-R)}$$

$$\frac{\pi.P \xrightarrow{\alpha} P'}{[x = x]\pi.P \xrightarrow{\alpha} P'} \qquad \text{(Match)}$$

$$\frac{P \xrightarrow{\alpha} P' \quad \alpha = \bar{x}(z) \implies x \notin fv(Q)}{P \mid Q \xrightarrow{\alpha} P' \mid Q} \qquad \text{(Par-L)}$$

$$\frac{Q \xrightarrow{\alpha} Q' \qquad \alpha = \bar{x}(z) \implies x \notin fv(P)}{P \mid Q \xrightarrow{\alpha} P \mid Q'} \qquad \text{(Par-R)}$$

$$\frac{P \xrightarrow{\alpha} P' \quad z \notin names(\alpha)}{\nu\ z\ \ P \xrightarrow{\alpha} \nu\ z\ \ P'} \quad \text{(Restriction)}$$

$$\frac{P \xrightarrow{\bar{x}y} P' \quad Q \xrightarrow{x(y)} Q'}{P \mid Q \xrightarrow{\tau} P' \mid Q'} \quad \text{(Comm-L)}$$

$$\frac{P \xrightarrow{x(y)} P' \quad Q \xrightarrow{\bar{x}y} Q'}{P \mid Q \xrightarrow{\tau} P' \mid Q'} \quad \text{(Comm-R)}$$

$$\frac{P \xrightarrow{\bar{x}z} P'}{\nu\ z\ P \xrightarrow{\bar{x}(z)} P'} \quad \text{(Open)}$$

$$\frac{P \xrightarrow{\bar{x}(z)} P' \quad Q \xrightarrow{x(z)} Q'}{P \mid Q \xrightarrow{\tau} \nu\ z\ (P' \mid Q')} \quad \text{(Close-L)}$$

$$\frac{P \xrightarrow{x(z)} P' \quad Q \xrightarrow{\bar{x}(z)} Q'}{P \mid Q \xrightarrow{\tau} \nu\ z\ (P' \mid Q')} \quad \text{(Close-R)}$$

$$\frac{P \xrightarrow{\alpha} P'}{!P \xrightarrow{\alpha} P' \mid !P} \quad \text{(Rep-Act)}$$

$$\frac{P \xrightarrow{\bar{x}y} P' \quad P \xrightarrow{x(z)} P'' \quad z \notin fv(P)}{!P \xrightarrow{\tau} (P' \mid P'') \mid !P} \quad \text{(Rep-Comm)}$$

$$\frac{P \xrightarrow{\bar{x}(z)} P' \quad P \xrightarrow{x(z)} P'' \quad z \notin fv(P)}{!P \xrightarrow{\tau} (\nu(P' \mid P'')) \mid !P} \quad \text{(Rep-Close)}$$

Fact For any P and α the set $\{P' \mid P \xrightarrow{\alpha} P'\}$ is finite.

Lemma 1 (Harmony)**.** $P \longrightarrow P' \iff P \xrightarrow{\tau}\equiv P'$

Fact $\longrightarrow$ is not image finite.
The primary issue is the ! operator – $!P$ may be expanded infinitely . Noting that $!P \equiv P \mid !P$, we can extend our notion of image finiteness to include 'up-to structural congruence'.

1.2 Reduction Bisimilarity

Definition 1 (Reduction bi-similarity)**.** Reduction bi-similarity is the largest symmetric relation S such that whenever $P\ S\ Q$ and $P \longrightarrow P'$ we have $Q(\longrightarrow S)\ P'$

This definition is somewhat unsatisfying, however. For instance, under this definition, $\bar{x}y.0$ and 0 are regarded as equivalent.

Definition 2 (Reduction congruence)**.** Two processes P and Q are reduction congruent if $C[P]$ and $C[Q]$ are reduction bi-similar for any context C.

P	Q	Congruent?	Context
$\bar{x}a.0$	$\bar{y}a.0$	No	$C = [\cdot] \mid x(z).0$
$\bar{a}x.0$	$\bar{a}y.0$	No	$C = [\cdot] \mid a(z).(\bar{z}b \mid y(w))$ $C[P] = \bar{a}x \mid a(z).(\bar{z}b \mid y(w))$ $\longrightarrow (\bar{x}b \mid y(w))$ $C[P] = \bar{a}y \mid a(z).(\bar{z}b \mid y(w))$ $\longrightarrow (\bar{y}b \mid y(w))$
$!\tau.0$	$(R \mid !\tau.0)$	Yes!	

We can't distinguish transitions and $!\tau.0$ can always step. This might feel familiar to "termination equivalence" in λ-calculus.

1.3 Observations

We write $P \downarrow x$ if P can perform input on x in the LTS. We write $P \downarrow \bar{x}$ if P can perform output on x in the LTS.

Definition 3 (Strong barbed bi-similarity)**.** Strong-barbed bi-similarity is the largest symmetric relation $\dot{\sim}$ such that whenever $P \dot{\sim} Q$:

- $P \downarrow \mu$ implies $Q \downarrow \mu$
- $P \xrightarrow{\tau} P'$ implies $Q \xrightarrow{\tau} \dot{\sim} P'$

Facts $\dot{\sim}$ is:

- an equivalence
- preserved under prefixes, sum, and restriction
- $\equiv \subseteq \dot{\sim}$ (Structural congruence implies $\dot{\sim}$)

Example

$$P = \quad \bar{x}y.\bar{a}\langle\rangle$$
$$Q = \quad \bar{x}y.0$$

$$C = \quad [\cdot] \mid x(z).0$$

Definition 4 (Strong-barbed congruence)**.** P and Q are strong barbed congruent, written $\simeq_c$ iff $C[P] \dot{\sim} C[Q]$ for any C.

Fact : $\simeq_c$ is the largest congruence included in $\dot{\sim}$, therefor $\equiv \subseteq \simeq_c$.

Lemma 2 (Context Lemma)**.** $P \simeq_c Q \iff P\sigma \mid R \dot{\sim} Q\sigma \mid R$ *for any* σ, R.

Definition 5 (Strong-barbed equivalence)**.** $P \simeq Q \overset{\triangle}{\iff} P \mid R \dot{\sim} Q|R$

Example The following example helps distinguish strong barbed congruence and strong barbed equivalence.

$$
\begin{aligned}
P = &\quad \bar{z} \mid a \\
Q = &\quad \bar{z}.a + a.\bar{z}
\end{aligned}
$$

It should be clear that P and Q are strong barbed equivalent – no matter what R we compose them with, at most two tau transitions are possible and the same observations (barbs) hold throughout.

However, P and Q are not strong barbed congruent. To see this, pick C to $x(z).[\cdot]])|\bar{x}\langle a\rangle$. Putting the hole $[\cdot]$ under a receive of z on x causes the substitution $[^a/_z]$ to be applied to in the process plugged into the hole during evaluation. This enables the two proceses composed in parallel on the P side to interact with each other while the Q side, no such interaction is possible. This also explains the critical role of the substitution in the Context Lemma – the context C constructed above essentially plays the role of the substitution $[^a/_z]$.