§ 66. The recursion theorem. Theorem XXVI. For any \( n \geq 0 \), let \( F(\zeta; x_1, \ldots, x_n) \) be a partial recursive functional, in which the function variable \( \zeta \) ranges over partial functions of \( n \) variables. Then the equation
\[
\zeta(x_1, \ldots, x_n) \equiv F(\zeta; x_1, \ldots, x_n)
\]
has a solution \( \varphi \) for \( \zeta \) such that any solution \( \varphi' \) for \( \zeta \) is an extension of \( \varphi \), and this solution \( \varphi \) is partial recursive.

Similarly, when \( \Psi \) are \( l \) partial functions and predicates,
\[
\zeta(x_1, \ldots, x_n) \equiv F(\zeta, \Psi; x_1, \ldots, x_n)
\]
has a solution \( \varphi \) for \( \zeta \) such that any solution \( \varphi' \) for \( \zeta \) is an extension of \( \varphi \), and this solution \( \varphi \) is partial recursive in \( \Psi \). (The first recursion theorem.)

Proof (for \( l = 0 \), \( n = 1 \)). Let \( \varphi_0 \) be the completely undefined function. Then introduce \( \varphi_1, \varphi_2, \varphi_3, \ldots \) successively by
\[
\varphi_1(x) \simeq F(\varphi_0; x), \quad \varphi_2(x) \simeq F(\varphi_1; x), \quad \varphi_3(x) \simeq F(\varphi_2; x), \ldots
\]
Since \( \varphi_0 \) is completely undefined, \( \varphi_1 \) is an extension of \( \varphi_0 \); then by Theorem XXI (a), \( \varphi_2 \) is an extension of \( \varphi_1 \), \( \varphi_3 \) of \( \varphi_2 \), etc. Let \( \varphi \) be the “limit function” of \( \varphi_0, \varphi_1, \varphi_2, \ldots \); i.e. for each \( x \), let \( \varphi(x) \) be defined if and only if \( \varphi_s(x) \) is defined for some \( s \), in which case its value is the common value of \( \varphi_s(x) \) for all \( s \geq \) the least such \( s \). Now:

(i) For each \( x \), \( \varphi(x) \simeq F(\varphi; x) \). For consider any \( x \). Suppose \( \varphi(x) \) is defined. Then for some \( s \), \( \varphi(x) \simeq \varphi_{s+1} \) [by definition of \( \varphi \) \( \simeq F(\varphi; x) \) [by definition of \( \varphi_{s+1} \simeq F(\varphi; x) \) [by Theorem XXI (a), since \( \varphi \) is an extension of \( \varphi_s \). Conversely, suppose \( F(\varphi; x) \) is defined; call its value \( \rho \). Since \( F \) is partial recursive, there is a system \( F \) of equations defining \( \varphi(\zeta; x) \) recursively from \( \zeta \), say with \( \rho \) as principal and \( \rho \) as given function letter; so now there is a deduction of \( f(x) = k \) from \( E \rho, F \). Let \( g(\mathbf{y}_1) = \mathbf{z}_1, \ldots, g(\mathbf{y}_p) = \mathbf{z}_p \) (where \( \mathbf{z}_i = \varphi(\mathbf{y}_i) \)).

(ii) If for each \( x \), \( \varphi(x) \simeq F(\varphi'; x) \), then \( \varphi \) is an extension of \( \varphi \). It will suffice to show by induction on \( \eta \) that, for each \( \varphi(\mathbf{y}) \) as defined, then \( \varphi(\mathbf{y}) \simeq \varphi_{\lambda}(\mathbf{y}) \). Basis: \( \lambda = 0 \). True vacuously. Ind. Step. Suppose for a given \( x \) that \( \varphi_{\lambda+1}(x) \) is defined. Then \( \varphi_{\lambda+1}(x) \simeq F(\varphi; x) \simeq F(\varphi'; x) \) [by Theorem XXI (a), since by hyp. \( \varphi \) is an extension of \( \varphi \).

(iii) If \( F \) defines \( F(\zeta; x) \) recursively from \( \zeta \), and \( E \) comes from \( F \) by substituting the principal function symbol \( f \) for the given function symbol \( g \), then \( E \) defines \( \varphi \) recursively. It will suffice to show that \( E \vdash f(x) = k \) if and only if \( \varphi(\mathbf{x}) \simeq k \) for some \( s \). We easily see that if \( \varphi(\mathbf{x}) = k \), then \( E \vdash f(x) = k \). For the converse, we show by induction on \( \eta \) that if there is a deduction of \( f(x) = k \) from \( E \) of height \( \eta \), then \( \varphi(\mathbf{x}) = k \) for some \( s \). The deduction can be altered if necessary, so that in each inference by \( R_2 \) with a minor premise of the form \( f(\mathbf{y}) = \mathbf{z} \) only one occurrence of \( f(\mathbf{y}) \) in the major premise is replaced by \( \mathbf{z} \) (Act 1). The occurrences of \( f \) in equations of the deduction can be classified in an evident manner into those which come from an occurrence of \( f \) in \( F \), and those which come via the substitution of \( f \) for an occurrence of \( g \) in \( F \). Now consider the inferences by \( R_2 \) with minor premise of the form \( f(\mathbf{y}) = \mathbf{z} \) in which the \( f \) of the part \( f(\mathbf{y}) \) replaced comes from a \( g \) in \( F \). Say there are \( p \) such inferences, the minor premises \( f(\mathbf{y}_1) = \mathbf{z}_1, \ldots, f(\mathbf{y}_p) = \mathbf{z}_p \) of which do not stand above other such premises. Each of these \( p \) premises occurred above the endequation of the given deduction before Act 1; so using the hypothesis of the induction, \( \mathbf{z}_1 \simeq \varphi_1(\mathbf{y}_1), \ldots, \mathbf{z}_p \simeq \varphi_p(\mathbf{y}_p) \), so \( \varphi(\mathbf{y}) \simeq \varphi(\mathbf{y}) \) where \( s = \max(s_1, \ldots, s_p) \). Now consider the tree remaining from the deduction after Act 1, when all the equations above \( f(\mathbf{y}) = \mathbf{z} \), \( \ldots, f(\mathbf{y}) = \mathbf{z} \) are removed (Act 2). In this tree, let each occurrence of \( f \) which (before Act 2) came from a \( g \) of \( F \) be changed back to \( g \) (Act 3). The \( f \)'s in question all occurred in the right members of equations, since \( g \) being the given
function symbol of F occurs in F only on the right; so no f is changed by
Act 3 in what was a minor premise for R2 before Act 3 or in the end-
equation f(x) = k. Finally, let the f's of f(y) = z_1, ..., f(y)_n = z_n be
changed to g (Act 4), which restores the inferences by R2 which Act 3
spoiled. The resulting tree is a deduction of f(x) = k from E_F. F. Hence

k \simeq f(p; x) \simeq q_{+1}(x).

**Example 1.** Consider the problem: to find a partial recursive function

\(\varphi\) such that

(a) \(\varphi(x) \simeq \varphi(x)\);

i.e. to solve the equation \(\zeta(x) \simeq \zeta(x)\) for \(\zeta\). Obviously any partial function
satisfies this equation. The partial function with the least range of
definition which satisfies is the completely undefined function. This is
the solution \(\varphi\) given by the theorem (with \(F(\zeta; x) \simeq \zeta(x)\)).

**Example 2.** To find a partial recursive function \(\varphi\) such that

(a) \(\varphi(x) \simeq \varphi(x)+1\);

i.e. to solve \(\zeta(x) \simeq \zeta(x)+1\) for \(\zeta\). Only the completely undefined partial
function satisfies this. This of course is the solution \(\varphi\) given by the theorem
(with \(F(\zeta; x) \simeq \zeta(x)+1\)).

**Example 3.** To find a function \(\varphi\) partial recursive in \(\chi\) such that

\[
\begin{cases}
\varphi(0) \simeq q, \\
\varphi(y') \simeq \chi(y', \varphi(y))
\end{cases}
\]

(Schema (Va) § 43). Only one function \(\varphi\) satisfies for a given \(\chi\), and we
already know by Theorem XVII (a) that it is partial recursive in \(\chi\).
However to see how the theorem applies, we rewrite (a) as

(b) \(\varphi(x) \simeq F(\varphi, \chi; x)\) where

\[
F(\zeta, \chi; x) \simeq \begin{cases}
q & \text{if } x = 0, \\
\chi(x-1, \chi(x-1)) & \text{if } x > 0
\end{cases}
\]

(equivalently,

\[
F(\zeta, \chi; x) \simeq \mu \omega \{x = 0 \& \omega = q \} \lor \{x > 0 \& \omega = \chi(x-1, \chi(x-1))\}.
\]

Since \(F(\zeta, \chi; x)\) is partial recursive (using Theorems XVII, XX (c); or
XVII, XVIII, XX (a)), by the theorem \(\varphi\) is partial recursive in \(\chi\).

**Example 4.** We give a new proof of Theorem XVIII (which proof in
various guises appeared in Kleene 1935, 1936, 1943). Let

\(\varphi(x) \simeq \mu y [\chi(x, y) = 0]\). Then \(\varphi(x) \simeq \varphi(x, 0)\) where

(a) \(\varphi(x, y) \simeq \mu \omega \geq y [\chi(x, \omega) = 0]\).

§ 66

**The Recursion Theorem**

But \(\varphi(x, y)\) is the partial function \(\varphi\) with the least range of definition such that

\[
\varphi(x, y) \simeq \begin{cases}
y & \text{if } \chi(x, y) = 0, \\
\varphi(x, y') & \text{if } \chi(x, y) \neq 0.
\end{cases}
\]

By Theorem XX (c) (with the first proof), the right side of (b) is of the
form \(F(\varphi, \chi; x, y)\) where \(F(\varphi, \chi; x, y)\) is partial recursive; so by the present
theorem \(\varphi(x, y)\), and hence \(\varphi(x)\), is partial recursive in \(\chi\).

**Discussion.** The theorem for \(l = 0\) asserts that we can impose any
relationship of the form

\[
\varphi(x_1, ..., x_n) \simeq F(\varphi; x_1, ..., x_n)
\]

expressing the ambiguous value \(\varphi(x_1, ..., x_n)\) of a function \(\varphi\) in terms
of \(\varphi\) itself and \(x_1, ..., x_n\) by methods already treated in the theory of
partial recursive functions; and conclude that the partial function with
the least range of definition which satisfies the relationship is partial
recursive.

Moreover the case of the theorem for \(l > 0\), in which

\[
\varphi(x_1, ..., x_n) \simeq F(\varphi, \psi; x_1, ..., x_n)
\]

is the relationship imposed, can be used to extend the body of the methods
available for use in further applications.

In our examples of special kinds of "recursion" (§§ 43, 46 and beginning
§ 55) the ambiguous function value \(\varphi(x_1, ..., x_n)\) was expressed in terms
of values of the same function for sets of arguments preceding the given
n-tuple \(x_1, ..., x_n\) in terms of some special ordering of the n-tuples. We
now have a general kind of "recursion", in which the value \(\varphi(x_1, ..., x_n)\)
can be expressed as depending on other values of the same function in
a quite arbitrary manner, provided only that the rule of dependence is
describable by previously treated effective methods.

The given "recursion" may now be ambiguous as a definition of an
ordinary (i.e. completely defined) number-theoretic function \(\varphi\), in the
sense that it is satisfied by more than one such function (Example 1, or (b)
in Example 4 when \(\chi(x, y)\) does not vanish for infinitely many values of \(y\).)
But now we choose as the solution which interests us that partial function
which is defined only when the recursion requires it to be. The given
"recursion" may be inconsistent as a definition of an ordinary function
(Example 2); again the difficulty is escaped now through the fact that it is
only a partial function which we are seeking as the solution. Both
these situations can arise when the F is general recursive (Examples 1
and 2). When F is incompletely defined, the recursion may also directly