Intuitionistic Semantics

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PLDG, April 17, 2013

Proof Rules of Intuitionistic Logic

$$\frac{\Gamma \vdash P \qquad \Gamma \vdash P \to Q}{\Gamma \vdash Q} \ (\to\text{-elim, modus ponens}) \qquad \frac{\Gamma, P \vdash Q}{\Gamma \vdash P \to Q} \ (\to\text{-intro})$$

$$\frac{\Gamma \vdash P \qquad \Gamma \vdash Q}{\Gamma \vdash P \land Q} \ (\land\text{-intro})$$

$$\frac{\Gamma \vdash P \land Q}{\Gamma \vdash P} \ (\land\text{-elim left}) \qquad \frac{\Gamma \vdash P \land Q}{\Gamma \vdash Q} \ (\land\text{-elim right})$$

$$\frac{}{\Gamma \vdash \top} \ (\text{unit}) \qquad \frac{}{\Gamma, P \vdash P} \ (\text{assum})$$

$$\frac{\Gamma \vdash P}{\Gamma \vdash P \lor Q} \ (\lor\text{-intro left}) \qquad \frac{\Gamma \vdash Q}{\Gamma \vdash P \lor Q} \ (\lor\text{-intro right})$$

$$\frac{\Gamma \vdash P \to R \qquad \Gamma \vdash Q \to R}{\Gamma \vdash P \lor Q \to R} \ (\lor\text{-elim})$$

$$\frac{\Gamma \vdash P \to \bot}{\Gamma \vdash \neg P} \ (\neg\text{-intro}) \qquad \frac{\Gamma \vdash \neg P}{\Gamma \vdash P \to \bot} \ (\neg\text{-elim}) \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash P} \ (\text{ex falso quodlibet})$$

A formula $\varphi \in \Phi$ is intuitionistically valid if there exists a proof tree whose root is labeled with the judgment $\vdash \varphi$ (that is, $\varnothing \vdash \varphi$).

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{x_1 : P}{P \lor Q}\,(\lor\mathrm{IL})
        \qquad
        \dfrac{x_0 : \neg(P \lor Q)}{P \lor Q \Rightarrow \bot}\,(\neg\mathrm{E})
      }{\bot}\,(\Rightarrow\mathrm{E})
    }{P \Rightarrow \bot}\,(\Rightarrow\mathrm{I}/x_1)
  }{\neg P}\,(\neg\mathrm{I})
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{x_2 : Q}{P \lor Q}\,(\lor\mathrm{IR})
        \qquad
        \dfrac{x_0 : \neg(P \lor Q)}{P \lor Q \Rightarrow \bot}\,(\neg\mathrm{E})
      }{\bot}\,(\Rightarrow\mathrm{E})
    }{Q \Rightarrow \bot}\,(\Rightarrow\mathrm{I}/x_2)
  }{\neg Q}\,(\neg\mathrm{I})
}{
  \dfrac{\neg P \land \neg Q}{\neg(P \lor Q) \Rightarrow \neg P \land \neg Q}\,(\Rightarrow\mathrm{I}/x_0)
}\,(\land\mathrm{I})
$$

The following rules are not permitted in intuitionistic proofs.

$$\frac{\Gamma, \neg P \vdash \bot}{\Gamma \vdash P} \text{ (reductio ad absurdum)} \qquad \frac{}{\Gamma \vdash P \vee \neg P} \text{ (tertium non datur)}$$

$$\frac{}{\Gamma \vdash \neg\neg P \rightarrow P} \text{ (double negation)} \qquad \frac{}{\Gamma \vdash ((P \rightarrow Q) \rightarrow P) \rightarrow P} \text{ (Peirce's law)}$$

They are intuitionistically interderivable.

Adding any one to intuitionistic logic gives classical propositional logic.

Realizability

We can annotate the intuitionistic rules with their realizers.

A realizer is a proof term generated by the grammar

$$t \ ::= \ x \mid \lambda x.t \mid s\,t \mid \langle s,\, t\rangle \mid \pi_0\,t \mid \pi_1\,t \mid [s,\, t] \mid \iota_0\,t \mid \iota_1\,t$$

Annotated Rules

$$\frac{\Gamma \vdash x : P \qquad \Gamma \vdash f : P \to Q}{\Gamma \vdash f\,x : Q} \qquad \frac{\Gamma, x : P \vdash t : Q}{\Gamma \vdash \lambda x.t : P \to Q}$$

$$\frac{\Gamma \vdash s : P \qquad \Gamma \vdash t : Q}{\Gamma \vdash \langle s,\, t\rangle : P \wedge Q} \qquad \frac{\Gamma \vdash t : P \wedge Q}{\Gamma \vdash \pi_0\,t : P} \qquad \frac{\Gamma \vdash t : P \wedge Q}{\Gamma \vdash \pi_1\,t : Q}$$

$$\frac{}{\Gamma \vdash \langle\rangle : \top} \qquad \frac{}{\Gamma, x : P \vdash x : P}$$

$$\frac{\Gamma \vdash t : P}{\Gamma \vdash \iota_0\,t : P \vee Q} \qquad \frac{\Gamma \vdash t : Q}{\Gamma \vdash \iota_1\,t : P \vee Q}$$

$$\frac{\Gamma \vdash f : P \to R \qquad \Gamma \vdash g : Q \to R}{\Gamma \vdash [f,\, g] : P \vee Q \to R}$$

$$\frac{\Gamma \vdash t : P \to \bot}{\Gamma \vdash t : \neg P} \qquad \frac{\Gamma \vdash t : \neg P}{\Gamma \vdash t : P \to \bot} \qquad \frac{\Gamma \vdash t : \bot}{\Gamma \vdash t : P}$$

A formula $\varphi \in \Phi$ is intuitionistically valid iff it has a closed realizer.

Heyting Algebra

A Heyting algebra is a bounded distributive lattice with an extra operation $\to$ satisfying

$$x \leq y \to z \quad \Leftrightarrow \quad x \wedge y \leq z$$

($\wedge, \vee$ have higher priority than $\to$, and $\to$ associates to the right).

It is axiomatized here with a pair of equational implications, but it is possible to give a purely equational axiomatization.

The pseudocomplement of x is $x \to \bot$, denoted $\neg x$.

HA : intuitionistic propositional logic :: BA : classical propositional logic

Some Theorems of Heyting Algebra

In any HA, the following are equivalent:

$$x \leq y \qquad x \vee y = y \qquad x \wedge y = x \qquad \top = x \to y$$

These expressions are all equal to $\top$ in any HA:

(MP) $x \wedge (x \to y) \to y$

($\wedge$I) $x \to y \to x \wedge y$

($\wedge$E) $(x \wedge y \to x) \wedge (x \wedge y \to y)$

(S) $(x \to y \to z) \to (x \to y) \to (x \to z)$

(K) $x \to y \to x$

Note that $x \vee \neg x \neq \top$ in general.

Heyting Algebras and Intuitionistic Logic

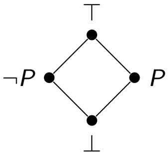
Heyting algebras provide a semantics for intuitionistic logic. A formula of propositional intuitionistic logic is a term in the language of Heyting algebras.

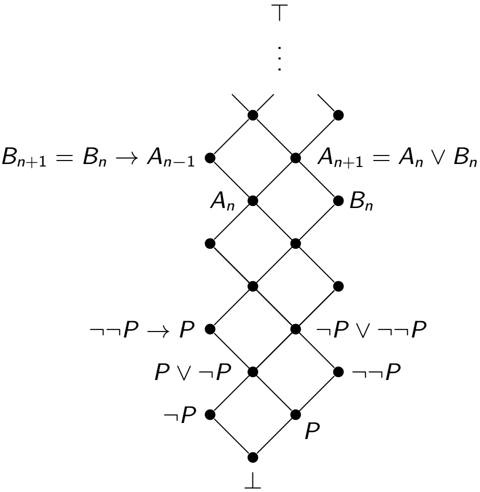
The set Φ modulo intuitionistic provability forms the free Heyting algebra on generators Φ_0.

Thus

- $\vdash \varphi$ iff $\varphi = \top$ in all Heyting algebras
- $\vdash \varphi \leftrightarrow \psi$ iff $\varphi = \psi$ in all Heyting algebras
- $\vdash \varphi \rightarrow \psi$ iff $\varphi \leq \psi$ in all Heyting algebras

The Rieger–Nishimura Lattice





The free BA on one generator. *The free HA on one generator.*

Completeness of Intuitionistic Logic for HA Semantics

Define

- $H \vDash \varphi$ if $\varphi = \top$ in the Heyting algebra H;
- $H \vDash \Gamma$ if $H \vDash \varphi$ for all $\varphi \in \Gamma$; and
- $\Gamma \vDash \varphi$ if for all H, $H \vDash \Gamma$ implies $H \vDash \varphi$.

Theorem (Completeness)

$\Gamma \vdash \varphi$ *iff* $\Gamma \vDash \varphi$.

Topological Spaces as Heyting Algebras

The open sets of any topological space form a complete Heyting algebra with

$$A \to B \;=\; (\sim A \cup B)^\circ,$$

where $^\circ$ denotes interior.

It is clearly a bounded distributive lattice under $\cup$ and $\cap$.

To show it is a Heyting algebra, we need only check that

$$A \subseteq (\sim B \cup C)^\circ \quad \Leftrightarrow \quad A \cap B \subseteq C.$$

The pseudocomplement is $\neg A = (\sim A)^\circ = \sim\overline{A}$, where $\overline{A}$ is the closure of A, and $\neg\neg A = (\overline{A})^\circ$. Thus $A \subseteq \neg\neg A$, but the inclusion in the opposite direction does not necessarily hold.

Topological Spaces as Heyting Algebras

Intuitionistic logic is sound and complete over this class of interpretations.

It is sound since all intuitionistic tautologies hold in all Heyting algebras.

It is complete, since every Heyting algebra has a topological representation (Priestley duality).

It can be shown that it is sufficient to interpret over a single Heyting algebra whose elements are the open subsets of $\mathbb{R}$. A formula is intuitionistically valid if its value is the entire real line under any valuation of the variables as open subsets of $\mathbb{R}$.

Kripke Models

A Kripke model is a triple $(S, \leq, \Vdash)$, where $\leq$ is a partial order on S and $\Vdash \subseteq S \times \Phi_0$ such that if $s \leq t$ and $s \Vdash P$, then $t \Vdash P$.

Extend $\Vdash$ to $S \times \Phi$:

$$
\begin{aligned}
s \Vdash \varphi \vee \psi &\Leftrightarrow s \Vdash \varphi \text{ or } s \Vdash \psi \\
s \Vdash \varphi \wedge \psi &\Leftrightarrow s \Vdash \varphi \text{ and } s \Vdash \psi \\
s \Vdash \varphi \rightarrow \psi &\Leftrightarrow (\forall t\ s \leq t \wedge t \Vdash \varphi \Rightarrow t \Vdash \psi) \\
s &\nVdash \bot \\
s &\Vdash \top
\end{aligned}
$$

It follows that

$$
s \Vdash \neg\varphi \Leftrightarrow (\forall t\ s \leq t \Rightarrow t \nVdash \varphi)
$$

Persistence

Lemma (Persistence)

If $s \leq t$ *and* $s \Vdash \varphi$*, then* $t \Vdash \varphi$*.*

Proof.

By induction. True for atomic formulas by definition.
Suppose $s \leq t$.

$$s \Vdash \varphi \vee \psi \;\Rightarrow\; s \Vdash \varphi \text{ or } s \Vdash \psi \;\Rightarrow\; t \Vdash \varphi \text{ or } t \Vdash \psi \;\Rightarrow\; t \Vdash \varphi \vee \psi$$

$$s \Vdash \varphi \wedge \psi \;\Rightarrow\; s \Vdash \varphi \text{ and } s \Vdash \psi \;\Rightarrow\; t \Vdash \varphi \text{ and } t \Vdash \psi \;\Rightarrow\; t \Vdash \varphi \wedge \psi$$

and $t \nVdash \bot$ and $t \Vdash \top$.
Finally, suppose $s \Vdash \varphi \to \psi$. For all u, if $t \leq u$ and $u \Vdash \varphi$, then $s \leq u$, therefore $u \Vdash \psi$. As u was arbitrary, $t \Vdash \varphi \to \psi$. □

Lemma (Density)

$s \Vdash \neg\neg\varphi \;\Leftrightarrow\; \forall t \geq s \; \exists u \geq t \;\; u \Vdash \varphi$.

Soundness and Completeness

Theorem

The following are equivalent:

1. $\Gamma \vdash \varphi$ *intuitionistically*
2. $\Gamma \vDash \varphi$ *in all Heyting algebras*
3. $\Gamma \vDash \varphi$ *in all topological spaces*
4. $\Gamma \Vdash \varphi$ *in all Kripke models*

Proof.

1⇒2: Heyting axioms model intuitionistic proof rules.
2⇒3: All topological spaces are Heyting algebras.
3⇒4: Given a Kripke model, let the sets $\varphi' = \{s \mid s \Vdash \varphi\}$ generate a topology.
4⇒1: Requires work! □

A set of formulas Γ is a prime theory if

- Γ is closed under $\vdash$
- $\varphi \vee \psi \in \Gamma \Rightarrow \varphi \in \Gamma$ or $\psi \in \Gamma$

Lemma (Extension theorem)

If $\Gamma \nvdash \varphi$, then there is a prime theory Γ' extending Γ such that $\Gamma' \nvdash \varphi$.

Proof.

Use Zorn's lemma. □

Soundness and Completeness

Lemma (Model existence lemma)

If $\Gamma \nvdash \varphi$, then there is a Kripke model K and $s \in K$ such that $s \Vdash \Gamma$ but $s \nVdash \varphi$.

Proof.

States of K are prime theories. □

Theorem (Completeness)

$\Gamma \vdash \varphi \;\Leftrightarrow\; \Gamma \Vdash \varphi$.

Relation to Classical Logic

Theorem (Glivenko's Theorem)

$\Gamma \vdash \varphi$ *classically iff* $\Gamma \vdash \neg\neg\varphi$ *intuitionistically.*

The translation $\varphi \mapsto \neg\neg\varphi$ works for propositional logic, but not for first-order logic.

The Gödel–Gentzen Translation

- $P' = \neg\neg P$, P atomic
- $(\varphi \wedge \psi)' = \varphi' \wedge \psi'$
- $(\varphi \vee \psi)' = \neg(\neg\varphi' \wedge \neg\psi')$
- $(\varphi \to \psi)' = \varphi' \to \psi'$
- $(\neg\varphi)' = \neg\varphi'$
- $(\forall x\ \varphi)' = \forall x\ \varphi'$
- $(\exists x\ \varphi)' = \neg\forall x\ \neg\varphi'$.

Alternatively,

- $(\varphi \vee \psi)' = \neg\neg(\varphi' \vee \psi')$
- $(\exists x\ \varphi)' = \neg\neg\exists x\ \varphi'$.

Note that $\vDash \varphi' \leftrightarrow \varphi$ classically.

Theorem

$\Gamma \vdash \varphi$ *classically iff* $\Gamma' \vdash \varphi'$ *intuitionistically.*