1. Give evidence semantics for each of the following propositional formulas in the Intuitionistic Propositional Calculus (IPC). For the starred examples, explain the formula as the statement of a programming problem or task.

* (a) $((A \& B) \Rightarrow C) \Rightarrow A \Rightarrow (B \Rightarrow C)$ (this has a name)

* (b) $(A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \& B) \Rightarrow C$ (so does this)

(c) $\neg (A \vee B) \Rightarrow \neg A \& \neg B$

(d) Recall that $(False \Rightarrow A)$ has evidence $\lambda(x. any(x))$,
$(\neg A \vee B) \Rightarrow (A \Rightarrow B)$

* (e) $(A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \vee B) \Rightarrow C)$

(f) $(A \Rightarrow B) \Rightarrow ((A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow C))$

* (g) $(A \Rightarrow B) \Rightarrow \neg B \Rightarrow \neg A$

(h) $\neg\neg (A \vee \neg A)$

2. What is an intuitive explanation for the invalidity of any of these formulas which is not valid? For valid ones (intuitionistically) provide the evidence.

(a) $\neg (A \& B) \Rightarrow \neg A \vee \neg B$ (b) $(\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)$

(c) $(A \Rightarrow B) \Rightarrow \neg A \vee B$ (d) $\neg\neg A \Rightarrow A$

(e) $\neg (A \Rightarrow B) \Rightarrow \neg B$

CS6110/6116 Problem Set 5 cont.

3. Prove $(A \Rightarrow A)$ using a Hilbert style proof from these two axiom schemas and the rule modus ponens

   1. $A \Rightarrow (B \Rightarrow A)$ as an axiom schema.
   2. formula (G) from Problem 1 treated as an axiom schema.

Modus Ponens $\dfrac{A \quad A \Rightarrow B}{B}$

Instances of axiom schemas arise by substituting any propositional formula for the propositional variables A, B, C. For example, $A \Rightarrow (A \Rightarrow A)$ is an instance of 1.

4. Prove the following formulas using Refinement Logic rules from Lecture 30, Wed April 11.

   (a) $P \vee (Q \& R) \Rightarrow (P \vee Q) \& (P \vee R)$ "distributivity"

   (b) $(P \Rightarrow \neg P) \Rightarrow \neg P$

   (c) $P \Rightarrow \neg\neg P$

   (d) $\neg P \Rightarrow (P \Rightarrow Q)$

   (e) $\neg\neg(P \vee \neg P)$

5. There will be one more problem with four or five parts after we examine First-Order Logic on Monday April 16. One part will be to give the evidence for

   (a) $\forall x.(A(x) \Rightarrow B(x)) \Rightarrow (\forall x.A(x) \Rightarrow \forall x.B(x))$

   (b) $\forall x.(A(x) \Rightarrow B(x)) \Rightarrow (\exists x\, A(x) \Rightarrow \exists x.B(x))$.

Also, for (c) and (d) below, give evidence if true, if not true explain why.

   (c) $(\exists x.A(x) \Rightarrow \exists x\, B(x)) \Rightarrow \exists x\,(A(x) \Rightarrow B(x))$

   (d) $\exists y(A(y) \Rightarrow B(y)) \Rightarrow \forall x.A(x) \Rightarrow \exists y.B(y)$