

1 Introduction

In Lecture 26, we proved that each term in the simply typed λ -calculus would never get stuck. Today, we want to show that it will actually terminate. This property is known as *strong normalization*.

Formally, we want to prove that if $\vdash e : \tau$, then $e \Downarrow$. We will prove this by induction, but we will need a fairly sophisticated induction hypothesis that takes both the typing and the reduction order into account. We cannot just do induction on the subterm relation. For example, even if e_1 and e_2 terminate, we cannot conclude that $e_1 e_2$ does: consider $e_1 = e_2 = \lambda x. xx$.

2 Church vs. Curry

We will prove this theorem in the pure simply-typed λ -calculus in Curry style. This differs from Church style in that the binding occurrence of a variable in a λ -abstraction is not annotated with its type.

Let α, β, \dots denote type variables, x, y, \dots term variables, σ, τ, \dots types, and d, e, \dots terms. In the Curry-style simply typed λ -calculus, terms and types are defined by

$$e ::= x \mid e_1 e_2 \mid \lambda x. e \qquad \tau ::= \alpha \mid \sigma \rightarrow \tau$$

and the typing rules are

$$\Gamma, x : \tau \vdash x : \tau \qquad \frac{\Gamma \vdash e : \sigma \rightarrow \tau \quad \Gamma \vdash d : \sigma}{\Gamma \vdash (e d) : \tau} \qquad \frac{\Gamma, x : \sigma \vdash e : \tau}{\Gamma \vdash (\lambda x. e) : \sigma \rightarrow \tau}$$

Note that in Church style, a closed term can have at most one type, but in Curry style, if it has any type at all, then it has infinitely many. For example, $\vdash \lambda x. x : ((\alpha \rightarrow \beta) \rightarrow \gamma) \rightarrow ((\alpha \rightarrow \beta) \rightarrow \gamma)$. In general, if $\vdash e : \tau$, then also $\vdash e : \tau'$, where τ' is any substitution instance of τ .

A term e is *typable* if there exists a type environment Γ and a type τ such that $\Gamma \vdash e : \tau$. One can show by induction that if $\Gamma \vdash e : \tau$, then $FV(e) \subseteq \text{dom } \Gamma$.

3 Strong Normalization

By the Church–Rosser theorem, normal forms are unique up to α -equivalence, so any two reduction strategies starting from the same term that terminate must yield the same result up to α -equivalence. However, there may be some strategies that terminate and some that do not.

A term is *strongly normalizing* (SN) if all β -reduction sequences starting from that term converge to a normal form; equivalently, if there is no infinite β -reduction sequence starting from that term. Our main theorem is

Theorem 1. *All typable terms are strongly normalizing.*

3.1 Ultra-Strong Normalization

We say that a term e is *ultra-strongly normalizing with respect to Γ and σ* and write $\Gamma \vdash_{USN} e : \sigma$ if

- (i) $\Gamma \vdash e : \sigma$

- (ii) for all $n \geq 0$, if σ is of the form $\sigma_1 \rightarrow \sigma_2 \rightarrow \cdots \rightarrow \sigma_n \rightarrow \tau$ and $\Gamma \vdash_{USN} e_i : \sigma_i$, $1 \leq i \leq n$, then $e \ e_1 \ e_2 \ \cdots \ e_n$ is SN.

A term e is *ultra-strongly normalizing* (USN) if it is ultra-strongly normalizing with respect to some Γ and σ .

The definition of the relation \vdash_{USN} may seem circular, but it is not: $\Gamma \vdash_{USN} e : \sigma$ is defined in terms of $\Gamma \vdash_{USN} e_i : \sigma_i$, where the σ_i are strict subexpressions of σ , so it is well-defined by structural induction on types.

Almost all the work we need to do is contained in the following lemma:

Lemma 2. *Let x_1, \dots, x_n be distinct variables. If*

- (i) $\Gamma, x_n : \sigma_n, \dots, x_1 : \sigma_1 \vdash e : \tau$,
- (ii) $\Gamma \vdash_{USN} d_i : \sigma_i$, $1 \leq i \leq n$, and
- (iii) $x_j \notin FV(d_i)$ for $j > i$,

then $\Gamma \vdash_{USN} e \{d_1/x_1\} \cdots \{d_n/x_n\} : \tau$.

Proof. Suppose the three premises (i)–(iii) hold. The proof is by induction on the structure of e .

Case 1 Variable x .

Case 1A $x = x_i$ for some i . We have $\tau = \sigma_i$ by assumption (i) and $x \{d_1/x_1\} \cdots \{d_n/x_n\} = d_i$ by assumption (iii). The desired conclusion is therefore $\Gamma \vdash_{USN} d_i : \sigma_i$, which follows from assumption (ii).

Case 1B $x \notin \{x_1, \dots, x_n\}$. We have $\Gamma \vdash x : \tau$ by assumption (i), and $x \{d_1/x_1\} \cdots \{d_n/x_n\} = x$. The desired conclusion is therefore $\Gamma \vdash_{USN} x : \tau$. We already have $\Gamma \vdash x : \tau$, so we need only show that $x \ e_1 \ \cdots \ e_m$ is SN for all appropriately typed USN terms e_i . But in any infinite β -reduction sequence starting from $x \ e_1 \ \cdots \ e_m$, every reduction must be inside one of the e_i , since there are no other β -redexes; therefore some e_i must contain an infinite subsequence. But this is impossible, since the e_i are USN.

Case 2 Application $e_1 \ e_2$. For some type σ ,

$$\begin{aligned} & \Gamma, x_n : \sigma_n, \dots, x_1 : \sigma_1 \vdash (e_1 \ e_2) : \tau \\ \Rightarrow & \Gamma, x_n : \sigma_n, \dots, x_1 : \sigma_1 \vdash e_1 : \sigma \rightarrow \tau \ \wedge \ \Gamma, x_n : \sigma_n, \dots, x_1 : \sigma_1 \vdash e_2 : \sigma \\ \Rightarrow & \Gamma \vdash_{USN} e_1 \{d_1/x_1\} \cdots \{d_n/x_n\} : \sigma \rightarrow \tau \ \wedge \ \Gamma \vdash_{USN} e_2 \{d_1/x_1\} \cdots \{d_n/x_n\} : \sigma \end{aligned} \quad (1)$$

by the induction hypothesis. By clause (i) in the definition of USN, this implies

$$\begin{aligned} & \Gamma \vdash e_1 \{d_1/x_1\} \cdots \{d_n/x_n\} : \sigma \rightarrow \tau \ \wedge \ \Gamma \vdash e_2 \{d_1/x_1\} \cdots \{d_n/x_n\} : \sigma \\ \Rightarrow & \Gamma \vdash (e_1 \ e_2) \{d_1/x_1\} \cdots \{d_n/x_n\} : \tau \end{aligned} .$$

This establishes clause (i) in the definition of USN for $e_1 \ e_2$. For clause (ii), we must show that if $\tau = \tau_3 \rightarrow \cdots \rightarrow \tau_m$ and if $\Gamma \vdash_{USN} e_i : \tau_i$ for $3 \leq i \leq m$, then

$$\begin{aligned} & (e_1 \ e_2) \{d_1/x_1\} \cdots \{d_n/x_n\} \ e_3 \ \cdots \ e_m \\ = & (e_1 \{d_1/x_1\} \cdots \{d_n/x_n\}) (e_2 \{d_1/x_1\} \cdots \{d_n/x_n\}) \ e_3 \ \cdots \ e_m \end{aligned} \quad (2)$$

is SN. But by (1),

$$\begin{aligned} \Gamma \vdash_{USN} e_1 \{d_1/x_1\} \cdots \{d_n/x_n\} : \sigma \rightarrow \tau_3 \rightarrow \cdots \rightarrow \tau_m \\ \Gamma \vdash_{USN} e_2 \{d_1/x_1\} \cdots \{d_n/x_n\} : \sigma \\ \Gamma \vdash_{USN} e_i : \tau_i, \quad 3 \leq i \leq m, \end{aligned}$$

thus (2) is SN. This proves that $\Gamma \vdash_{USN} (e_1 \ e_2) \{d_1/x_1\} \cdots \{d_n/x_n\} : \tau$.

Case 3 Abstraction $\lambda x.e$. We can assume without loss of generality that $\lambda x.e$ has been α -converted so that $x \notin FV(d_i)$ and $x \neq x_i$ for any i , $1 \leq i \leq n$. Instead of x , let us call this bound variable x_{n+1} . Then for some σ_{n+1} , we have

- (i) $\Gamma, x_n : \sigma_n, \dots, x_1 : \sigma_1 \vdash (\lambda x_{n+1}.e) : \sigma_{n+1} \rightarrow \tau$,
- (ii) $\Gamma \vdash_{USN} d_i : \sigma_i$, $1 \leq i \leq n$, and
- (iii) $x_j \notin FV(d_i)$ for $j > i$ (including $j = n+1$),

and we wish to show $\Gamma \vdash_{USN} (\lambda x_{n+1}.e) \{d_1/x_1\} \cdots \{d_n/x_n\} : \sigma_{n+1} \rightarrow \tau$.

Starting from assumption (i), we have

$$\begin{aligned} \Gamma, x_n : \sigma_n, \dots, x_1 : \sigma_1 \vdash (\lambda x_{n+1}.e) : \sigma_{n+1} \rightarrow \tau \\ \Rightarrow \Gamma, x_n : \sigma_n, \dots, x_1 : \sigma_1, x_{n+1} : \sigma_{n+1} \vdash e : \tau \\ \Rightarrow \Gamma, x_{n+1} : \sigma_{n+1}, x_n : \sigma_n, \dots, x_1 : \sigma_1 \vdash e : \tau. \end{aligned}$$

If d_{n+1} is any term such that $\Gamma \vdash_{USN} d_{n+1} : \sigma_{n+1}$, then by the induction hypothesis we have both

$$\Gamma, x_{n+1} : \sigma_{n+1} \vdash_{USN} e \{d_1/x_1\} \cdots \{d_n/x_n\} : \tau \quad (3)$$

$$\Gamma \vdash_{USN} e \{d_1/x_1\} \cdots \{d_{n+1}/x_{n+1}\} : \tau. \quad (4)$$

For clause (i) in the definition of USN, starting from (3), we have

$$\begin{aligned} \Gamma, x_{n+1} : \sigma_{n+1} \vdash e \{d_1/x_1\} \cdots \{d_n/x_n\} : \tau \\ \Rightarrow \Gamma \vdash \lambda x_{n+1}. (e \{d_1/x_1\} \cdots \{d_n/x_n\}) : \sigma_{n+1} \rightarrow \tau \\ \Rightarrow \Gamma \vdash (\lambda x_{n+1}.e) \{d_1/x_1\} \cdots \{d_n/x_n\} : \sigma_{n+1} \rightarrow \tau \quad \text{since } x_{n+1} \notin FV(d_i). \end{aligned}$$

For clause (ii), we wish to show that if in addition to the assumptions (i)–(iii) above, $\tau = \sigma_{n+2} \rightarrow \cdots \rightarrow \sigma_m \rightarrow \rho$ and $\Gamma \vdash_{USN} d_i : \sigma_i$, $n+1 \leq i \leq m$, then

$$\begin{aligned} (\lambda x_{n+1}.e) \{d_1/x_1\} \cdots \{d_n/x_n\} d_{n+1} \cdots d_m \\ = (\lambda x_{n+1}. (e \{d_1/x_1\} \cdots \{d_n/x_n\})) d_{n+1} \cdots d_m \end{aligned}$$

is SN. Consider any infinite reduction sequence starting from this term. We know that $e \{d_1/x_1\} \cdots \{d_n/x_n\}$ is SN by (3), and we know that the d_i are SN by assumption, $n+1 \leq i \leq m$. Therefore, eventually a head reduction must be performed:

$$\begin{aligned} (\lambda x_{n+1}. (e \{d_1/x_1\} \cdots \{d_n/x_n\})) d_{n+1} \cdots d_m \\ \xrightarrow{*} (\lambda x_{n+1}. (e \{d_1/x_1\} \cdots \{d_n/x_n\})') d'_{n+1} \cdots d'_m \\ \rightarrow (e \{d_1/x_1\} \cdots \{d_n/x_n\})' \{d'_{n+1}/x_{n+1}\} d'_{n+2} \cdots d'_m. \end{aligned}$$

But we could have done the head reduction initially:

$$\begin{aligned}
& (\lambda x_{n+1}. (e\{d_1/x_1\} \cdots \{d_n/x_n\})) d_{n+1} \cdots d_m \\
& \rightarrow e\{d_1/x_1\} \cdots \{d_n/x_n\} \{d_{n+1}/x_{n+1}\} d_{n+2} \cdots d_m \\
& \xrightarrow{*} (e\{d_1/x_1\} \cdots \{d_n/x_n\})' \{d'_{n+1}/x_{n+1}\} d'_{n+2} \cdots d'_m,
\end{aligned}$$

leading to an infinite reduction sequence from $e\{d_1/x_1\} \cdots \{d_n/x_n\} \{d_{n+1}/x_{n+1}\} d_{n+2} \cdots d_m$. But this contradicts (4). \square

Proof of Theorem 1. Any typable term is USN: take $n = 0$ in Lemma 2. Any term that is USN is SN: take $n = 0$ in the definition of USN. \square