We saw that the semantics of the **while** command are a fixed point. We also saw that intuitively, the semantics are the limit of a series of approximations capturing a finite number of iterations of the loop, and giving a result of $\perp$ for greater numbers of iterations. In order to take a limit, we need greater structure, which led us to define partial orders. But ordering is not enough.

1 Complete partial orders (CPOs)

Least upper bounds Given a partial order $(S, \sqsubseteq)$, and a subset $B \subseteq S$, y is an *upper bound* of B *iff* $\forall x \in B. x \sqsubseteq y$. In addition, y is a *least upper bound iff* y is an upper bound and $y \sqsubseteq z$ for all upper bounds z of B. We may abbreviate "least upper bound" as LUB or lub. We notate the LUB of a subset B as $\bigsqcup B$. We may also make this an infix operator, writing $\bigsqcup_{i \in 1..m} x_i = x_1 \sqcup \ldots \sqcup x_m = \bigsqcup \{x_i\}_{i \in 1..m}$. This is also known as the *join* of elements $x_1, \ldots, x_m$.

Chains A *chain* is a pairwise comparable sequence of elements from a partial order (i.e., elements $x_0, x_1, x_2 \ldots$ such that $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \ldots$). For any finite chain, its LUB is its last element (e.g., $\bigsqcup x_i = x_n$). Infinite chains (ω-chains, i.e. indexed by the natural numbers) may also have LUBs.

Complete partial orders A *complete partial order* (CPO)[1] is a partial order in which every chain has a least upper bound. Note that the requirement that this hold for *every* chain is trivial for finite partial orders—it is infinite chains that can cause trouble.

Some examples partial orders that are complete or not complete:

- Any finite partial order is complete: any infinite chain must have a highest element.
- $(\mathcal{P}^S, \subseteq)$: complete. The LUB of a chain is just the union of all sets in the chain.
- $(\mathbb{N}, \leq)$: not complete. The chain $0 \leq 1 \leq 2 \leq \ldots$ has no upper bound.
- $(\mathbb{N} \cup \{\infty\}, \leq)$ Here, ∞ is the LUB for any infinite chain that does not repeat.
- $([0,1], \leq)$ where $[0,1]$ is the closed continuum: complete. Note that making the continuum open at the top – $[0,1)$ – would cause this to no longer be a CPO, since there would be no LUB for infinite chains such as $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$
- $(S, =)$: all discrete CPOs are complete. The only infinite chains are of the sort $x_i \sqsubseteq x_i \sqsubseteq x_i \ldots$, of which x_i is itself a LUB.
- $(S_\perp, \sqsubseteq)$ where $S_\perp$ is flat: complete. Any chain must have a highest element which is either $\perp$ or $\lfloor x \rfloor$ for $x \in S$.

Even if $(S, \sqsubseteq)$ is a CPO, $(S, \sqsupseteq)$ is not necessarily a CPO. Consider $((0,1], \leq)$, which is a CPO. Reversing its binary relation yields $((0,1], \geq)$ which is not a CPO, just as $([0,1), \leq)$ above was not.

A CPO D can also have a least element, written $\perp$, such that $\forall x \in D.\ \perp \sqsubseteq x$. We call a CPO with such an element a *pointed CPO*. Winskel instead uses *CPO with bottom*. A flat CPO is pointed.

[1]Mathematicians often write this in lower case: "cpo".

2 Least fixed points of functions

Recall that at the end of the last lecture we were attempting to define the least fixed point operator *fix* over the domain $(\Sigma \to \Sigma_\perp)$ so that we could determine calculate fixed points of $F : (\Sigma \to \Sigma_\perp) \to (\Sigma \to \Sigma_\perp)$. It was unclear, however, what the "least" fixed point of this domain would be—how is one function from states to states "less" than another? We've now developed the theory to answer that question.

We define the ordering of states by *information content*: $\sigma \sqsubseteq \sigma'$ *iff* σ gives less (or at most as much) information than σ'. Non-termination is defined to provide less information than any other state: $\forall\sigma \in \Sigma. \sqsubseteq \sigma$. In addition, we have that $\sigma \sqsubseteq \sigma$. No other pairs of states are deemed comparable. The lifted set of possible states $\Sigma_\perp$ is a flat CPO (a lifted discrete CPO), which is pointed and complete.

3 Functions

We are now ready to define an ordering relation on functions. Functions will be ordered by a *pointwise ordering* on their results. Given a CPO E, a domain set D (it need not be a CPO), $f \in D \to E$, and $g \in D \to E$:

$$f \sqsubseteq_{D\to E} g \overset{def}{\iff} \forall x \in D.\ f(x) \sqsubseteq_E g(x)$$

Note that we are defining a new partial order over $D \to E$, and that this CPO is pointed if E is pointed, since $\perp_{D\to E} = \lambda x \in D.\perp_E$.

As an example, consider two functions $\mathbb{Z} \to \mathbb{Z}_\perp$:

$$\begin{aligned} f &= \lambda x \in \mathbb{Z}\,.\,\mathbf{if}\ x = 0\ \mathbf{then}\ \perp\ \mathbf{else}\ x \\ g &= \lambda x \in \mathbb{Z}\,.\,x \end{aligned}$$

We conclude $f \sqsubseteq g$ because $f(x) \sqsubseteq g(x)$ for all x; in particular, $f(0) = \perp \sqsubseteq 1 = g(0)$.

If E is a CPO, then the function space $D \to E$ is also a CPO. We show that given a chain of functions $f_1 \sqsubseteq f_2 \sqsubseteq f_3 \ldots$, the function $\lambda d \in D\,.\,\bigsqcup_{n\in\mathbb{N}} f_n(d)$ is a least upper bound for this chain. Consider any function g that is an upper bound for all the f_n. In that case, we have:

$$\begin{aligned} & \forall n \in \mathbb{N}.\ \forall d \in D.\ f_n(d) \sqsubseteq g(d) \\ \iff\ & \forall d \in D.\ \forall n \in \mathbb{N}.\ f_n(d) \sqsubseteq g(d) \end{aligned}$$

Because the f_n form a chain, so do the $f_n(d)$, and because E is a CPO, it has a least upper bound that is necessarily less than the upper bound $g(d)$:

$$\begin{aligned} \implies\ & \forall d \in D.\ (\bigsqcup_{n\in\mathbb{N}} f_n(d)) \sqsubseteq g(d) \\ \iff\ & \forall d \in D.\ (\bigsqcup_{n\in\mathbb{N}} f_n)(d) \sqsubseteq g(d) \\ \iff\ & \bigsqcup_{n\in\mathbb{N}} f_n \sqsubseteq g \end{aligned}$$

Therefore, $D \to E$ is a CPO under the pointwise ordering.

4 Back to **while**

It's now time to unify our dual understanding of the denotation of **while** as both a limit and a fixed point.

We previously suggested the denotation of **while** is both:

$$\begin{aligned}\mathcal{C}[\![\textbf{while}\ b\ \textbf{do}\ c]\!] &= \mathit{fix}(F)\\ &= \text{limit of } F^n(\bot)\end{aligned}$$

However, we did not know how to define the *fix* operator over the range of F, nor did we have a definition for the least fixed point of F to take as its limit. CPOs now give us the necessary machinery.

We assert that:

$$\mathcal{C}[\![\textbf{while}\ b\ \textbf{do}\ c]\!] = \bigsqcup_{n\in\mathbb{N}} F^n(\bot)$$

As an example to give us confidence that this is the correct definition, we see that:

$$\begin{aligned}\mathcal{C}[\![\textbf{while true do skip}]\!] &= \bigsqcup_{n\in\mathbb{N}} F^n(\bot)\\ &= \bot_{\Sigma\to\Sigma_\bot}\\ &= \lambda\sigma\in\Sigma.\bot\end{aligned}$$

5 Monotonicity

As we begin to construct a proof that this denotation is correct, we want to show that this limit, or LUB, is a least fixed point of F. That is, we want to show that

$$\bigsqcup_{n\in\mathbb{N}} F^n(\bot)$$

is the least solution to

$$x = F(x)$$

.

However, this is not true for some F, such the following:

$$\begin{aligned}F(x) = {} & \text{if } x=\bot \text{ then } 1 \text{ else}\\ & \text{if } x = 1 \text{ then } 0 \text{ else } \bot\end{aligned}$$

Although 0 is clearly a fixed point of this F, $F^n(\bot)$ is not a chain (the elements cycle between $\bot$, 1, and 0), and so we cannot take its least upper bound.

Requires that F is *monotonic* fixes this problem:

Definition: Let $(D, \sqsubseteq)$ be a CPO, $F : D \to D$ a function. F is *monotonic* if

$$\forall x, y \in D.\ x \sqsubseteq y \implies F(x) \sqsubseteq F(y).$$

Claim: If $(D, \sqsubseteq, \bot)$ is a pointed CPO and $F : D \to D$ is monotonic then the elements $F^n(\bot)$ form an increasing chain in D:

$$\bot \sqsubseteq F(\bot) \sqsubseteq F^2(\bot) \sqsubseteq \dots$$

Proof: Since $\perp$ is the least element of D, we have

$$\perp \sqsubseteq F(\perp).$$

Monotonicity of F gives

$$\forall n \in \mathbb{N}.\ F^n(\perp) \sqsubseteq F^{n+1}(\perp) \Rightarrow F^{n+1}(\perp) \sqsubseteq F^{n+2}(\perp).$$

The result follows by induction.

6 Continuity

Monotonicity guarantees that the elements $F^n(\perp)$ are a chain and hence that we can find a LUB. But it doesn't mean we have a fixed point. Consider a monotonic but non-continuous F defined over the pointed CPO $(\mathbb{R} \cup \{-\infty, \infty\}, \leq)$:

$$F(x) = \textbf{if}\, x < 0\, \textbf{then}\, \tan^{-1}(x)\, \textbf{else}\, 1$$

This function is monotonic, and its least fixed point is 1. However,

$$\begin{array}{rcl} F^1(\perp) & = & \tan^{-1}(-\infty) = -\dfrac{\pi}{2} \\ F^2(\perp) & = & \tan^{-1}(-\dfrac{\pi}{2}) = -1 \\ F^2(\perp) & = & \tan^{-1}(-1) \approx -0.78 \end{array}$$

For $x < 0$, $F(x) > x$ and $F(x) < 0 :\ F^n(\perp)$ is a chain that approaches 0 arbitrarily closely: its LUB is 0. But $F(0) = 1$, so the LUB is not a fixed point! The least fixed point of this monotonic function is actually $1 = F(1)$. The problem with this function F is that it is not continuous at 0. In general, we will look for a (weaker) form of *continuity* in F for *fix* to guarantee that the LUB formula gives us a (least) fixed point.

Notice that if $F : D \to D$ is monotonic and $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \ldots$ is a chain in D, then $F(x_0) \sqsubseteq F(x_1) \sqsubseteq F(x_2) \sqsubseteq \ldots$ is also a chain in D. This permits the following definition.

Definition: Let $(D, \sqsubseteq)$ be a CPO, $F : D \to D$ a monotonic function. F is *continuous* if for every chain

$$x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \ldots$$

in D, F preserves the LUB operator:

$$\bigsqcup_{n\in\mathbb{N}} F(x_n) \quad = \quad F(\bigsqcup_{n\in\mathbb{N}} x_n).$$

7 The Fixed-Point Theorem

We will now show that the properties of monotonicity and continuity allow us to compute the least fixed point as desired.

Claim: Let $(D, \sqsubseteq)$ be a pointed CPO, and let $F : D \to D$ be a monotonic, continuous function. Then $\bigsqcup_{n\in\mathbb{N}} F^n(\perp)$ is a fixed point of F.

Proof: By continuity of F,

$$F(\bigsqcup_{n\in\mathbb{N}} F^n(\perp)) = \bigsqcup_{n\in\mathbb{N}} F(F^n(\perp))$$

Applying F,

$$= \bigsqcup_{n\in\mathbb{N}} F^{n+1}(\perp)$$

Reindexing,

$$= \bigsqcup_{n=1,2,\dots} F^n(\perp)$$

By definition of $\perp$,

$$= \perp \sqcup \bigsqcup_{n=1,2,\dots} F^n(\perp)$$

And, finally, absorbing the join with $\perp$ into the big join,

$$= \bigsqcup_{n\in\mathbb{N}} F^n(\perp)$$

We now know that monotonicity and continuity guarantee that $\bigsqcup_{n\in\mathbb{N}} F^n(\perp)$ is a fixed point of F. We also want $\bigsqcup_{n\in\mathbb{N}} F^n(\perp)$ to be the *least* fixed point of F. To show this, we must prove that $y = F(y) \Rightarrow \bigsqcup_{n\in\mathbb{N}} F^n(\perp) \sqsubseteq y$. We can actually prove something even stronger.

Definition: Let $(D, \sqsubseteq)$ be a CPO, $F : D \to D$ a function. $x \in D$ is a *prefixed* point of F if $F(x) \sqsubseteq x$.

Notice that every fixed point of F is also a prefixed point. As a consequence, if a fixed point of F is the least prefixed point of F, it is also the least fixed point of F.

Claim: Let $(D, \sqsubseteq, \perp)$ be a pointed CPO. For any monotonic continuous function, $F : D \to D$, $\bigsqcup_{n\in\mathbb{N}} F^n$ is the least prefixed point of F.

Proof: Suppose y is a prefixed point of F. By definition of $\perp$,

$$\perp \sqsubseteq y$$

Taking F of both sides,

$$F(\perp) \sqsubseteq F(y) \sqsubseteq y$$

Inductively, for all $n \geq 0$,

$$F^n(\perp) \sqsubseteq y$$

Because y is an upper bound for all the $F^n(\perp)$, it must be at least as large as their least upper bound:

$$\bigsqcup_{n\in\mathbb{N}} F^n(\perp) \sqsubseteq y$$

We have now proven:

The Fixed-Point Theorem: Let $(D, \sqsubseteq, \perp)$ be a pointed CPO. For any monotonic continuous function, $F : D \to D$, $\bigsqcup_{n\in\mathbb{N}} F^n$ is the least fixed point of F.

8 An instance of the FPT

We have actually encountered an instance of the fixed-point theorem before. Recall lecture 6, when we defined the set of all elements derivable in some rule system to be the least fixed point of the rule operator, R. Our proof in that case was an instantiation of the fixed point theorem on the CPO consisting of all subsets of a set, ordered by set inclusion:

$$R = F$$

$$\emptyset = \bot$$

$$\bigcup = \bigsqcup$$

$$\subseteq = \sqsubseteq$$

The tricky part of the earlier proof corresponded to showing that R is a continuous operator, which was true because we only allow inference rules with a finite number of premises.