CS 611
Advanced Programming Languages

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Lecture 34
Type inference & ML polymorphism
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Type inference

Simple typed language:

$e ::= x \mid b \mid \lambda x{:}\tau\,.\,e \mid e_1\, e_2 \mid e_1 \oplus e_2$
$\quad \mid \text{if } e_0 \text{ then } e_1 \text{ else } e_2 \mid \text{let } x{=}e_1 \text{ in } e_2$
$\quad \mid \text{rec } y{:}\tau_1{\rightarrow}\tau_2.(\lambda x.e)$

$\tau ::= \text{unit} \mid \text{bool} \mid \text{int} \mid \tau_1{\rightarrow}\tau_2$

- Question: Do we really need to write type declarations?

$e ::= \ldots \mid \lambda x.e \mid \ldots \mid \text{rec } y.(\lambda x.e)$

Typing rules

$e ::= x \mid b \mid \lambda x.e \mid e_1\, e_2 \mid e_1 \oplus e_2$
$\quad \mid \text{if } e_0 \text{ then } e_1 \text{ else } e_2 \mid \text{let } x{=}e_1 \text{ in } e_2 \mid \text{rec } y.\lambda x.e$

$$\frac{\Gamma, x{:}\tau \vdash e : \tau'}{\Gamma \vdash \lambda x\,.\,e : \tau{\rightarrow}\tau'}$$

$$\frac{\Gamma, y{:}\tau{\rightarrow}\tau', x{:}\tau \vdash e : \tau'}{\Gamma \vdash \text{rec } y.\lambda x\,.\,e : \tau{\rightarrow}\tau'}$$

Problem: how does type checker construct proof?

Guess τ ?

Example

```
let square = λz.z*z in
  (λf.λx.λy.
     if (f x y)
          then (f (square x) y)
          else (f x (f x y)))
```

What is the type of this program?

Example

```
let square = λz.z*z in
  (λf.λx.λy.
     if (f x y)
          then (f (square x) y)
          else (f x (f x y)))
```

z : int
s, square : int→int
f : $\tau_x{\rightarrow}\tau_y{\rightarrow}$bool
y : τ_y =bool
x : τ_x = int

Answer:
(int→bool→bool)→int→bool→bool

Type inference

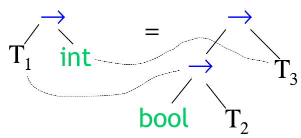
- Goal: reconstruct types even after erasure
- Idea: run ordinary type-checking algorithm, generate *type equations* on *type variables*

$$\frac{\dfrac{\dfrac{\text{f:T2, x:T5} \vdash \text{f : int}\rightarrow\text{T6} \qquad \text{f:T2, x:T5} \vdash \text{1 : int}}{\text{f:T2, x:T5} \vdash \text{f 1 : T6}}}{\dfrac{\text{f:T2} \vdash \lambda\text{x. f 1 : T1 (=T5}\rightarrow\text{T6)}}{\vdash \lambda\text{f. }\lambda\text{x. f 1 : T2}\rightarrow\text{T1}}} \qquad \dfrac{\dfrac{}{\text{y:T3} \vdash \text{y : T4}}\ \text{(T3=T4)}}{\vdash (\lambda\text{y.y) : T2} \quad \text{(T2=T3}\rightarrow\text{T4)}}}{\vdash (\lambda\text{f.}\lambda\text{x. (f 1)) (}\lambda\text{y.y) : T1}}$$

T2 = T3→T4, T3 = T4, T1 =T5→T6, T2 = int→T6

Unification

- How to solve equations?
- Idea: given equation $\tau_1 = \tau_2$, *unify* type expressions to solve for variables in both
- Example: $T_1 \rightarrow \text{int} = (\text{bool} \rightarrow T_2) \rightarrow T_3$
- Result: *substitution* $T_1 \mapsto \text{bool} \rightarrow T_2,\ T_3 \mapsto \text{int}$



Robinson's algorithm (1965)

- *Unification* produces *weakest substitution* that equates two trees
  - $T_1 \rightarrow \text{int} = (\text{bool} \rightarrow T_2) \rightarrow T_3$ equated by any $T_1 \mapsto \text{bool} \rightarrow T_2,\ T_3 \mapsto \text{int},\ T_2 \mapsto \tau$
  - **Defn.** S_1 is weaker than S_2 if $S_2 = S_3 \circ S_1$ for S_3 a non-identity substitution
- **Unify**(E) where E is set of equations gives weakest equating substitution: define recursively

Unify$(T = \tau, E)$ = **Unify**$(E\{\tau/T\}) \circ [T \mapsto \tau]$
  (if $T \notin FTV[\![\tau]\!]$)

Rest of algorithm

Unify$(T = \tau, E)$ = **Unify**$(E\{\tau/T\}) \circ [T \mapsto \tau]$
  (if $T \notin FTV[\![\tau]\!]$)
Unify$(\varnothing) = \varnothing$
Unify$(B = B, E)$ = **Unify**(E)
  Unify$(B_1 = B_2, E)$ = ?
Unify$(T = T, E)$ = **Unify**(E)
Unify$(\tau_1 \rightarrow \tau_2 = \tau_3 \rightarrow \tau_4, E)$
  = **Unify**$(\tau_1 = \tau_3,\ \tau_2 = \tau_4, E)$
Termination?

Type inference algorithm

- $\mathcal{R}(e, \Gamma, S) = \langle \tau, S' \rangle$ means "Reconstructing the type of e in typing context Γ with respect to substitution S yields type τ, identical or stronger substitution S' *or*

"S' is weakest substitution no weaker than than S such that $S'(\Gamma) \vdash e : S'(\tau)$"

Define: **Unify**(E, S) = **Unify**$(SE) \circ S$
  – solve substituted equations E and fold in new substitutions

Inductive defn of inference

- $\mathcal{R}(e, \Gamma, S) = \langle \tau, S' \rangle \Leftrightarrow$ "S' is weakest substitution stronger than (or same as) S such that $S'(\Gamma) \vdash e : S'(\tau)$"
- **Unify**(E, S) = **Unify**$(SE) \circ S$

$\mathcal{R}(n, \Gamma, S) = \langle \text{int}, S \rangle$ $\mathcal{R}(\#t, \Gamma, S) = \langle \text{bool}, S \rangle$
$\mathcal{R}(x, \Gamma, S) = \langle \Gamma(x), S \rangle$
$\mathcal{R}(e_1\, e_2, \Gamma, S)$ = let $\langle T1, S1 \rangle = \mathcal{R}(e_1, \Gamma, S)$ in
  let $\langle T2, S2 \rangle = \mathcal{R}(e_2, \Gamma, S1)$ in
  $\langle T_f$, **Unify**$(T1 = T2 \rightarrow T_f, S2) \rangle$
$\mathcal{R}(\lambda x.e, \Gamma, S)$ = let $\langle T1, S1 \rangle = \mathcal{R}(e, \Gamma[x \mapsto T_f], S)$ in
  $\langle T_f \rightarrow T1, S1 \rangle$

where T_f is "fresh" (not mentioned anywhere in e, Γ, S)

Example

$\mathcal{R}((\lambda x.x)\ 1, \varnothing, \varnothing)$ =
 let $\langle T1, S1 \rangle = \mathcal{R}(\lambda x.x, \varnothing, \varnothing)$ in
  let $\langle T2, S2 \rangle = \mathcal{R}(1, \varnothing, S1)$ in
   $\langle T3$, **Unify**$(T1 \rightarrow T3 = T2, S2) \rangle$

$\mathcal{R}(\lambda x.x, \varnothing, \varnothing)$ = let $\langle T1, S1 \rangle = \mathcal{R}(x, \Gamma[x \mapsto T4], \varnothing)$ in
  $\langle T4 \rightarrow T1, S1 \rangle$
  $= \langle T4 \rightarrow T4, \varnothing \rangle$

= let $\langle T2, S2 \rangle = \mathcal{R}(1, \varnothing, \varnothing)$ in
  $\langle T3$, **Unify**$(T2 \rightarrow T3 = T4 \rightarrow T4, \varnothing) \rangle$
= $\langle T3$, **Unify**$(\text{int} \rightarrow T3 = T4 \rightarrow T4, \varnothing) \rangle$
= $\langle T3$, **Unify**$(\text{int} = T4,\ T3 = T4, \varnothing) \rangle$
= $\langle T3$, **Unify**$(T3 = \text{int}, [T4 \mapsto \text{int}]) \rangle$
= $\langle T3, [T3 \mapsto \text{int},\ T4 \mapsto \text{int}] \rangle$

Polymorphism

$\mathcal{R}(\lambda x.x, \varnothing, \varnothing)$ = let $\langle T1, S1\rangle = \mathcal{R}(x, \Gamma[x \mapsto T4], \varnothing)$ in
$\langle T4 \to T1, S1\rangle$
$= \langle T4 \to T4, \varnothing\rangle$

- Reconstruction algorithm doesn't solve type fully… opportunity!
- $\lambda x.x$ can have type T4→T4 for any T4
  - polymorphic (= "many shape") term
  - Could reuse same expression multiple places in program, with different types:

let id = (λx.x) in … (f id) … (g x id) … id

Polymorphic types

- Type expression may have some unsolved type identifiers after type reconstruction
- Type T4→T4 is a *type schema* that can be instantiated with any T4 to make a type
- ML idea: let can bind identifiers to polymorphic terms
  - typing context Γ maps variable either to
    - type τ or
    - type schema $\forall T_1,\ldots,T_n.\ \tau$ where $FTV(\tau) \subseteq \{T_1,\ldots,T_n\}$
- Can still do type inference! (ML)

Typing rules

$\Gamma \in Var \rightharpoonup \sigma$

$\sigma ::= \tau \mid \forall T_1,\ldots,T_n.\ \tau$

$\Delta = \{T_1,\ldots,T_n\}$ (Δ : set of legal type variables)

$\Delta \vdash \tau$ (judgment: τ is well formed)

$\Delta ; \Gamma \vdash e : \tau$

$$\frac{}{\Delta ; \Gamma, x{:}\tau \vdash x : \tau} \qquad \frac{\Delta ; \Gamma, x{:}\tau \vdash e : \tau' \quad \Delta \vdash \tau \quad \Delta \vdash \tau'}{\Delta ; \Gamma \vdash \lambda x.e : \tau \to \tau'}$$

$$\frac{\Delta \vdash \tau_i \quad {}^{\forall i \in 1..n}}{\Delta ; \Gamma, x{:}(\forall T_1,\ldots,T_n.\ \tau) \vdash x : \tau\{\tau_i / T_i\ {}^{\forall i \in 1..n}\}}$$

More typing rules

$$\frac{\Delta ; \Gamma \vdash e_1 : \tau \to \tau' \quad \Delta ; \Gamma \vdash e_2 : \tau \to \tau' \quad \Delta \vdash \tau, \tau'}{\Delta ; \Gamma \vdash e_1\ e_2 : \tau'}$$

$$\frac{\Delta \cup \{T_1,\ldots,T_n\} ; \Gamma \vdash e_1 : \tau \quad \Delta \cup \{T_1,\ldots,T_n\} \vdash \tau \quad \Delta ; \Gamma, x : \forall T_1,\ldots,T_n.\tau \vdash e_2 : \tau' \quad \Delta \vdash \tau'}{\Delta ; \Gamma \vdash \text{let } x = e_1 \text{ in } e_2 : \tau'}$$

$$\frac{T_1 ; \varnothing \vdash (\lambda x.x) : T_1 \to T_1 \quad \dfrac{\varnothing; \text{id} : \forall T_1.T_1 \to T_1 \vdash \text{id} : \text{int} \to \text{int}}{\varnothing; \text{id} : \forall T_1.T_1 \to T_1 \vdash \text{id } 2 : \text{int}}}{\varnothing;\varnothing \vdash \text{let id} = (\lambda x.x) \text{ in id } 2 : \text{int}}$$

Algorithm $\mathcal{W}$ (Milner)

- Infers types in language with let-bound type schemas! (& letrec too)
- $\mathcal{W}(e, \Gamma, S) = \langle \tau, S'\rangle$ gives type, subst S′ as before, (but Γ can map vars to type schemas)

$\mathcal{W}(x, \Gamma, S)$ = case $\Gamma(x)$ of
$\tau \Rightarrow \langle \tau, S\rangle$
$\mid \forall T_1,\ldots,T_n.\tau \Rightarrow \langle \tau\{T_{fi}/T_i\}, S\rangle$

$\mathcal{W}$(letrec $x = e_1$ in e_2, Γ, S) =
let $\Gamma' = \Gamma[x \mapsto T_f]$ in let $\langle T1, S1\rangle = \mathcal{W}(e_1, \Gamma', S)$ in
let S2=**Unify**($\{T_f = T1\}$, S1) in
let $\Gamma'' = \Gamma[x \mapsto \text{Generic}(T1, \Gamma, S2)]$ in
$\mathcal{W}(e_2, \Gamma'', S2)$

Generic(τ,Γ,S) = $\forall T_1,\ldots,T_n.S\tau$
where $\{T_1,\ldots,T_n\} = FTV(S\tau) - FTV(S\Gamma)$